

Review for Exam I

1. Differentiate the following functions

(a) $y = \sin^2(x)$

$$y'(x) = 2 \sin(x) \cos(x)$$

(b) $f(y) = \sec(3y)$

$$f'(y) = 3 \sec(3y) \tan(3y)$$

(c) $g(x) = x^2 \cos(5x)$

$$g'(x) = 2x \cos(5x) - 5x^2 \sin(5x)$$

(d) $y(x) = \arctan(5x)$

$$y'(x) = \frac{5}{1 + 25x^2}$$

(e) $f(x) = \int_{\pi}^{4x^3} u \sin(5u) du$

$$f'(x) = 48x^5 \sin(20x^3)$$

(f) $k(x) = \int_x^{\ln x} u^3 g(u) du$

$$k'(x) = \frac{(\ln x)^3 g(\ln x)}{x} - x^3 g(x)$$

2. Integrate the following

(a) $\int \frac{6}{x^7} dx$

$$= \frac{-1}{x^6} + c$$

(b) $\int_2^6 \sqrt{4x+1} dx$

$$= \frac{1}{6} (4x+1)^{3/2} \Big|_2^6 = \frac{1}{6} (25^{3/2} - 9^{3/2}) = \frac{49}{3}$$

(c) $\int \cos^4(x) \sin(x) dx$

$$u = \cos x \quad du = -\sin x dx$$

$$\text{answer: } = -\frac{1}{5} \cos^5 x + c$$

(d) $\int \frac{dx}{x \ln x}$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$\text{answer: } = \ln(\ln(x)) + c$$

$$(e) \int_0^1 (2x-1)^{100} dx$$

$$= \frac{1}{202} (2x-1)^{101} \Big|_0^1 = \frac{1}{101}$$

$$(f) \int x e^{-2x^2} dx$$

$$u = -2x^2 \quad \frac{-1}{4} du = x dx$$

$$\text{answer: } \frac{-1}{4} e^{-2x^2} + c$$

$$(g) \int \frac{5x}{1+4x^4} dx$$

$$u = 2x^2 \quad \frac{5}{4} du = 5x dx$$

$$\text{answer: } \frac{5}{4} \arctan(2x^2) + c$$

$$(h) \int x^2 \sqrt{3x-1} dx$$

$$u = 3x-1 \quad \frac{1}{3} du = dx \quad x = \frac{u+1}{3}$$

$$= \frac{1}{3} \int \left(\frac{u+1}{3} \right)^2 u^{1/2} du = \frac{1}{27} \int (u^2 + 2u + 1) u^{1/2} du$$

$$\text{answer: } \frac{1}{27} \left(\frac{2}{7} (3x-1)^{7/2} + \frac{4}{5} (3x-1)^{5/2} + \frac{2}{3} (3x-1)^{3/2} \right) + c$$

$$(i) \int_{-3}^3 \sqrt{9-x^2} dx$$

area of half-circle radius 3: answer is $\frac{9}{2}\pi$

$$(j) \int_{-3}^3 x \sqrt{9-x^2} dx$$

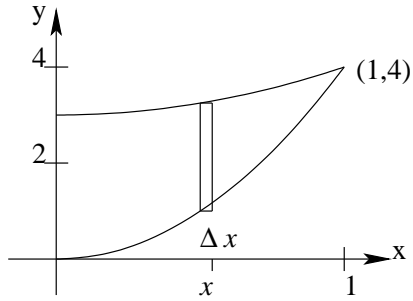
odd function over symmetric limits: answer is 0.

by computation = $-\frac{1}{3} (9-x^2)^{3/2} \Big|_{x=-3}^{x=3} = 0$.

$$(k) \int_{-3}^3 x^3 \sqrt{9-x^2} dx$$

odd function over symmetric limits: answer is 0.

3. (a) Graph $y = 4x^2$ and $y = x^2 + 3$ on the same pair of coordinate axes and calculate the area between them.

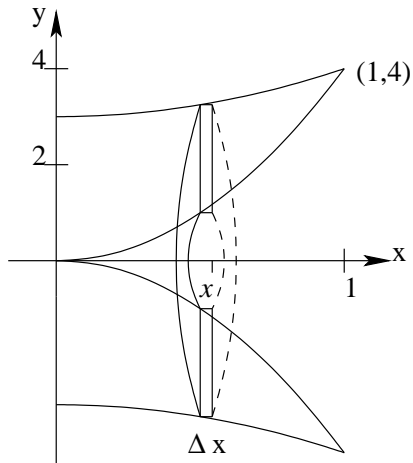


Length of rectangle at x is $(x^2 + 3 - 4x^2)$. Width of rectangle at x is Δx .

Note: This is only the right half of the drawing. By symmetry we can double this area value to include the left side.

$$\text{Area} = 2 \int_0^1 (x^2 + 3 - 4x^2) dx = -x^3 + 3x \Big|_0^1 = 2.$$

- (b) Now consider this same region rotated about the x -axis. Determine the volume of the solid formed.

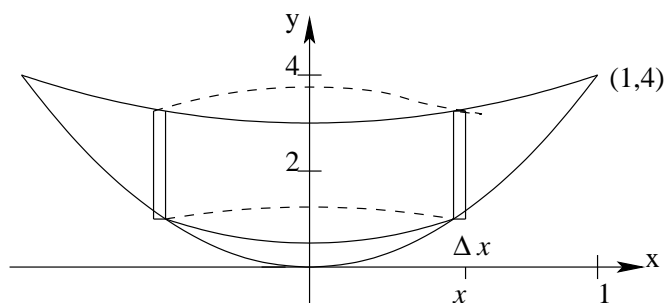


Volume of outside disk at x is $\pi(x^2 + 3)^2 \Delta x$.

Volume of inside disk at x is $\pi(4x^2)^2 \Delta x$.

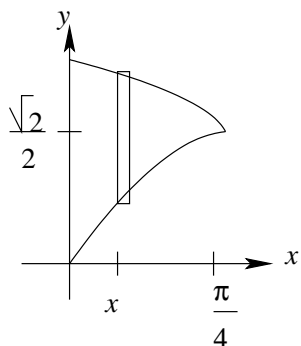
$$\text{Volume of solid is } \int_0^1 (x^4 + 6x^2 + 9 - 16x^4) dx$$

- (c) Now consider this same region rotated about the y -axis. Determine the volume of the solid formed.



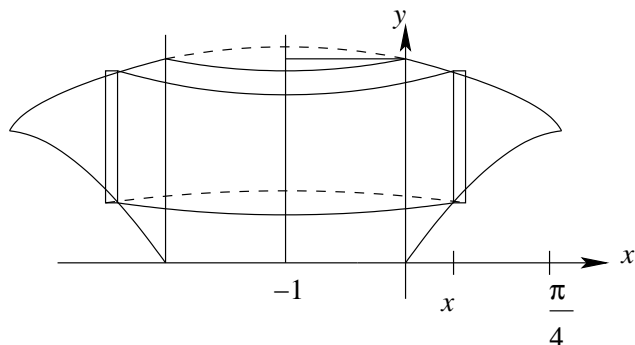
Circumference of shell at x is $2\pi x$.
 Length of shell at x is $x^2 + 3 - 4x^2$.
 Thickness of shell at x is Δx .
 Volume of shell at x is $2\pi x(x^2 + 3 - 4x^2)\Delta x$.
 Volume of solid is $2\pi \int_0^1 (x^3 + 3x - 4x^3) dx$

4. (a) Graph $y = \sin(x)$ and $y = \cos(x)$ on the same pair of coordinate axes and calculate the area between them on $[0, \pi/4]$.



Length of rectangle at x is $(\cos(x) - \sin(x))$. Width at x is Δx .
 Area of region is $\int_0^{\pi/4} (\cos(x) - \sin(x)) dx$.

- (b) Now consider this same region rotated about the line $x = -1$. Set up the integral for the volume of the solid formed.



Circumference of shell at x is $2\pi(x + 1)$.

Length of shell at x is $(\cos(x) - \sin(x))$.

Thickness of shell at x is Δx .

Volume of shell at x is $2\pi(1 + x)(\cos(x) - \sin(x))\Delta x$.

Volume of solid is $2\pi \int_0^{\pi/4} (1 + x)(\cos(x) - \sin(x)) dx$.

- (c) Now consider this same region rotated about the line $y = -1$. Set up the integral for the volume of the solid formed.

Volume of the outside disk is $\pi(\cos(x) + 1)^2 \Delta x$.

Volume of the inside disk is $\pi(\sin(x) + 1)^2 \Delta x$.

Volume of the solid is $\pi \int_0^{\pi/4} [(\cos(x) + 1)^2 - (\sin(x) + 1)^2] dx$

5. Determine the value of b so that the average value of the function $f(x) = x - 1$ is 8 on $[0, b]$.

$$\text{We have } 8 = \frac{1}{b - 0} \int_0^b (x - 1) dx.$$

$$\text{So that } 8b = \left. \frac{1}{2}x^2 - x \right|_0^b.$$

Which gives the equation $8b = \frac{1}{2}b^2 - b$, and $b = 18$.

6. A cone is formed by rotating the line $y = 3x - 4$, $0 \leq x \leq 4$, about the y -axis. If the cone is filled with water, determine the work done to empty the water out from the top of the cone. Use feet and pounds as units. See figure below, left.

Answer:

Add the work elements to lift each individual disk of water out of the cone.

The distance to raise a disk from the level y to the top is $8 - y$.

The weight of the disk at level y is $62.4lb/ft^3$ (volume of disk at level y).

The volume of the disk at level y is $\pi \left(\frac{y + 4}{3} \right)^2 \Delta y$.

The work element at level y is $W_y = 62.4\pi \left(\frac{y+4}{3}\right)^2 (8-y)\Delta y$.

$$\text{Work} = \frac{62.4\pi}{9} \int_{-4}^8 (y+4)^2(8-y) dy$$

7. A bucket of water with capacity 3 cubic feet is filled to the top and hangs from a cable 20 feet into a well. The cable weighs .5 lb per foot. If the bucket spills 0.5 lbs per foot raised, determine the work to raise the bucket and cable to the top of the well.

Answer:

Add the work to raise the bucket to the work to raise the cable.

Bucket work: The bucket starts out with a weight (force) of $62.4 \cdot 3 = 187.2$ lb.

Take the distance the bucket has to be moved and break it up into n pieces each Δy length and consider the work to raise the work that Δy distance.

The weight (force) at any height is $187.2 - 0.5y$ and the distance Δy .

Therefore, the work element is $(187.2 - 0.5y)\Delta y$ and

Work to raise bucket is $\int_0^{20} (187.2 - 0.5y) dy$.

Cable work: Take the cable and break it into n "links" each of Δy length.

Then the force (weight) of the cable is $0.5\Delta y$ pounds.

The distance that link has to be raised is y units to the top.

Therefore, the work element is $0.5y\Delta y$

Work to raise cable is $\int_0^{20} 0.5y dy$.

8. A swimming pool at ground level has width 20 feet and length 30 feet. Its base 10 feet below has width 20 feet and length 20 feet. Its front and back sides are isocelus trapezoids. See figure below, right. Determine the work necessary to empty a full swimming pool taking the water out of the top.

Answer:

This time we are moving a "slab" up to the top of the pool and out. Consider the work to raise each individual slab of water.

The distance a slab at level y has to be moved is $(10 - y)$.

The weight of the slab is $62.4 \cdot (\text{volume of a rectangular slab})$.

The width of any slab is 20 ft. and the thickness is Δy .

Half the length of the slab is the x -coordinate along the line connecting the points $(10, 0)$ and $(15, 10)$. Which is the line $y = 2x - 20$.

The x -coordinate at y then is $\frac{y+20}{2}$.

The length of the slab at y is then twice this which is $y + 20$.
 Therefore, the work element at y is $62.4(20)(y + 20)(10 - y)\Delta y$. and

$$\text{Work} = 62.4(20) \int_0^{10} (y + 20)(10 - y) dy.$$

