Review Exam II

1. Integrate the following

   (a) $\int \sec^4(3x)\tan^2(3x)\,dx$
   You have a du by using $\sec^2(3x)dx$ and use the identity $\sec^2(3x) = 1 - \tan^2(3x)$.
   You then have $\int (\tan^2(3x) - \tan^4(3x))\underline{\sec^2(3x)dx} = \frac{1}{9}\tan^3(3x) - \frac{1}{15}\tan^5(3x) + c$.

   (b) $\int \sin^3(2x)\,dx$
   One factor of the $\sin(2x)dx$ is your du and use the identity $\sin^2(2x) = 1 - \cos^2(2x)$.
   You then have $\int (1 - \cos^2(2x))\underline{\sin(2x)dx} = -\frac{1}{2}\cos(2x) + \frac{1}{6}\cos^3(2x) + c$.

   (c) $\int \frac{dx}{(9+4x^2)^{3/2}}$
   Use trig substitution. $\frac{3}{2}\tan\theta = x$ so that $dx = \frac{3}{2}\sec^2\theta\,d\theta$.
   Substitute and get: $\int \frac{\frac{3}{2}\sec^2\theta d\theta}{27\sec^3\theta} = \int \frac{1}{18}\cos\theta\,d\theta = \frac{1}{18}\sin\theta + c = \frac{2x}{\sqrt{9+4x^2}} + c$.

   (d) $\int \frac{x-3}{\sqrt{4-x^2}}\,dx$
   Use trig substitution. $2\sin\theta = x$ so that $dx = 2\cos\theta\,d\theta$.
   Substitute and get: $\int \frac{2\sin\theta - 3}{2\cos\theta} 2\cos\theta\,d\theta = \int (2\sin\theta - 3)\,d\theta$
   $= -2\cos\theta - 3\theta + c = \sqrt{4-x^2} - 3\arcsin(\frac{x}{2}) + c$.

   (e) $\int x\cos(3x)\,dx$
   Integrate by parts: $u = x \quad dV = \cos(3x)\,dx$; $du = dx \quad V = \frac{1}{3}\sin(3x)$
   $= \frac{1}{3}x\sin(3x) - \int \frac{1}{3}\sin(3x)\,dx = \frac{1}{3}\sin(3x) + \frac{1}{9}\cos(3x) + c$.

   (f) $\int x^3\ln(x)\,dx$
   Integrate by parts: $u = \ln(x) \quad dV = x^3\,dx$; $du = \frac{1}{x}\,dx \quad V = \frac{1}{4}x^4$

$$= \frac{1}{4}x^4 \ln(x) - \int \frac{1}{4}x^3\,dx = \frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + c.$$

(g) $\displaystyle\int \frac{1}{x^2 - x}\,dx$

Partial Fractions: $\displaystyle\int \frac{1}{x(x-1)}\,dx = \int \left(-\frac{1}{x} + \frac{1}{x-1}\right)dx.$

$\displaystyle= -\ln|x| + \ln|x-1| + c = \ln\left|\frac{x-1}{x}\right| + c.$

(h) $\displaystyle\int \frac{x^3+4}{x^2-4}\,dx$

Partial Fractions but first reduce. $\displaystyle\int \left(x + \frac{4x}{(x-2)(x+2)}\right)dx = \int \left(x + \frac{1}{x+2} + \frac{3}{x-2}\right)dx.$

$\displaystyle= \frac{1}{2}x^2 + \ln|x+2| + 3\ln|x-2| + c.$

(i) $\displaystyle\int \frac{x^2}{\sqrt{x-1}}\,dx$

Rationalize by letting $u = \sqrt{x-1}$. Then $u^2 = x - 1$ and $2u\,du = dx$

$\displaystyle\int \frac{(u^2+1)^2}{u}2u\,du = 2\int (u^4 + 2u^2 + 1)\,du$

$\displaystyle= \frac{2}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + c.$

(j) $\displaystyle\int x\sqrt{9+x^2}\,dx$

Use u-substitution. Let $u = 9 + x^2$. Then $\frac{1}{2}du = x\,dx$ and

$\displaystyle\int \frac{1}{2}\sqrt{u}\,du = \frac{1}{3}(9+x^2)^{3/2} + c.$

(k) $\displaystyle\int \frac{x^5}{\sqrt{1-x^3}}\,dx$

Rationalize by letting $u = \sqrt{1-x^3}$. Then $u^2 = 1 - x^3$ and $-\frac{2}{3}u\,du = x^2\,dx$.

Now $x^5\,dx = x^3 \cdot x^2\,dx = (1-u^2)(\frac{-2}{3}u\,du)$.

$\displaystyle-\frac{2}{3}\int \frac{(1-u^2)u}{u}\,du = -\frac{2}{3}\int (1-u^2)\,du = -\frac{2}{3}\sqrt{1-x^3} - \frac{2}{9}(1-x^3)^{3/2} + c.$

(l) $\displaystyle\int \tan x\,dx$

Use u-substitution: $\displaystyle\tan x = \frac{\sin x}{\cos x}.$

$$\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c.$$

2. Tell wether the integral converges or diverges. If it converges, give the limit. If it diverges, show why.

   (a) $\displaystyle\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx$

   $\displaystyle\lim_{t\to 9}\int_1^t \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{t\to 9}\left[\frac{3}{2}(x-9)^{2/3}\right]_1^t = \lim_{t\to 9}\left[\frac{3}{2}(t-9)^{2/3} - 4\right] = -6.$

   Therefore, the integral converges.

   (b) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

   $\displaystyle\lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^t = \lim_{t\to\infty}\left[\frac{1}{2}(\ln t)^2 - 0\right] = \infty.$

   Therefore, the integral diverges.

   (c) $\displaystyle\int_0^\infty xe^{-2x}\,dx$

   $\displaystyle\lim_{t\to\infty}\int_0^t xe^{-2x}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^t = \lim_{t\to\infty}\left[\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} - (0 - \frac{1}{4})\right] =$

   $\frac{1}{4}$. Therefore, the integral converges.

   (d) $\displaystyle\int_1^3 \frac{1}{(x-3)^2}\,dx$

   $\displaystyle\lim_{t\to 3}\int_1^t \frac{1}{(x-3)^2}\,dx = \lim_{t\to 3}\left[-\frac{1}{x-3}\right]_1^t = \lim_{t\to 3}\left[-\frac{1}{t-3} + \frac{1}{2}\right] = \infty.$

   Therefore, the integral diverges.

3. Solve the following differential equations

   (a) $y' = xe^y, \qquad y(2) = 0.$

$$\begin{aligned}
\frac{dy}{dx} &= xe^y\\
e^{-y}\,dy &= x\,dx\\
\int e^{-y}\,dy &= \int x\,dx\\
-e^{-y} &= \frac{1}{2}x^2 + c\\
y &= -\ln(-\frac{1}{2}x^2 + c)\\
y &= -\ln(-\frac{1}{2}x^2 + 3)
\end{aligned}$$

(b) $x\,dx - 2y\sqrt{x^2+1}\,dy = 0, \qquad y(0) = 1.$

$$\begin{aligned} 2y\,dy &= \frac{x}{\sqrt{x^2+1}}\,dx \\ \int 2y\,dy &= \int \frac{x}{\sqrt{x^2+1}}\,dx \\ y^2 &= (x^2+1)^{1/2} \end{aligned}$$

4. A tank initially contains 60 gallons of brine in which 15 lb of salt is dissolved. Brine containing 0.5 lb of salt per gallon flows into the tank at a constant rate of 3 gal/min. The concentration of brine and salt in the tank is kept uniform by stirring, and the brine is drawn off at a rate of 3 gal/min. Find the amount of salt in the tank after 10 min.

   Draw the picture and label it with the correct units, First!

   Let $y(t)$ be the amount of salt in the tank at time t. Then $\frac{dy}{dt}$ is the change in the amount of salt as time changes.

$$\begin{aligned} \frac{dy}{dt} &= \text{rate-in} - \text{rate-out} \\ \frac{dy}{dt} &= \frac{3}{2} - \frac{y}{20} \\ \frac{dy}{dt} &= -\frac{1}{20}(y-30) \\ \int \frac{dy}{y-30} &= \int -\frac{1}{20}\,dt \\ \ln(y-30) &= -\frac{1}{20}t + c \\ y &= 30 + Ae^{-t/20} \\ y &= 30 - 15e^{-t/20} \end{aligned}$$

   After 10 minutes there are $30 - 15e^{-1/2}$ pounds of salt in the tank.

5. Integration Bee is March 21. To sign up now, e-mail Dr Rubin at rubin@math.pitt.edu