PRACTICE6 Name_____________________________________ Section______

1. A ball rolls along a marked table and its position at any time can be determined by the parametric equations: $x(t) = t^3 - t^2$ and $y(t) = t^3 - 3t$. Determine $\frac{dy}{dx}$ when $t = 3$.

$$\frac{dx}{dt} = 3t^2 - 2t\Big|_{t=3} = 21, \text{ and } \frac{dy}{dt} = 3t^2 - 3\Big|_{t=3} = 24.$$

$$\frac{dy}{dx}\Big|_{t=3} = \frac{dy/dt}{dx/dt}\Big|_{t=3} = \frac{24}{21} = \frac{8}{7}$$

2. The paths $\vec{r_1}(t) = \langle t, t^2\rangle$ and $\vec{r_2}(t) = \langle \sin(t), \sin(2t)\rangle$ intersect when $t = 0$. Determine the angle of intersection by determining the angle between their tangent vectors.

$$\left\langle \frac{dx_1}{dt}, \frac{dy_1}{dt} \right\rangle\Big|_{t=0} = \langle 1, 2t\rangle|_{t=0} = \langle 1, 0\rangle$$

$$\left\langle \frac{dx_2}{dt}, \frac{dy_2}{dt} \right\rangle\Big|_{t=0} = \langle \cos(t), 2\cos(2t)\rangle|_{t=0} = \langle 1, 2\rangle$$

angle between curves: $\theta = \arccos\left(\frac{\langle 1,0\rangle \cdot \langle 1,2\rangle}{|\langle 1,0\rangle||\langle 1,2\rangle|}\right) = \arccos\left(\frac{1}{1\cdot\sqrt{5}}\right)$

3. Determine the angle of intersection of the paths $\vec{r}(t) = \langle t^2 + t + 2, \sin(\sqrt{3}\,t)\rangle$ and $\vec{s}(t) = \langle 2e^{\sqrt{3}\,t}, 2t\rangle$ as they cross at the time $t = 0$ through the point $(2, 0)$.

$$\left\langle \frac{dx_1}{dt}, \frac{dy_1}{dt} \right\rangle\Big|_{t=0} = \langle 2t+1, \sqrt{3}\cos(\sqrt{3}t)\rangle\Big|_{t=0} = \langle 1, \sqrt{3}\rangle$$

$$\left\langle \frac{dx_2}{dt}, \frac{dy_2}{dt} \right\rangle\Big|_{t=0} = \langle 2\sqrt{3}e^{\sqrt{3}t}, 2\rangle\Big|_{t=0} = \langle 2\sqrt{3}, 2\rangle$$

angle between curves: $\theta = \arccos\left(\frac{\langle 1,\sqrt{3}\rangle \cdot \langle 2\sqrt{3},2\rangle}{|\langle 1,\sqrt{3}\rangle||\langle 2\sqrt{3},2\rangle|}\right) = \arccos\left(\frac{4\sqrt{3}}{2\cdot 4}\right) = \frac{\pi}{6}$

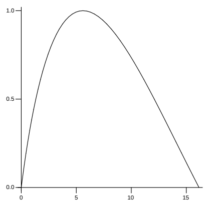
4. Determine the arc length of the path $x(t) = e^t + e^{-t}$, $y(t) = 5 - 2t$ on $0 \le t \le 4$.

$$\left(\frac{dx}{dt}\right)^2 = (e^t - e^{-t})^2 = e^{2t} - 2 + e^{-2t} \quad\text{and}\quad \left(\frac{dy}{dt}\right)^2 = 4$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$$

$$\int_0^4 \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^4 (e^t + e^{-t})\, dt = \left(e^t - e^{-t}\right)_0^4 = e^4 - e^{-4}$$

5. Determine the area bounded by the curve $x(t) = t^2 + 2t$, $y(t) = \sin(t)$ on $0 \le t \le \pi$.



$$\int_0^{\pi} y\, dx = \int_0^{\pi} \sin(t)(2t+2)\, dt$$
$$= (-2t\cos(t) + 2\sin(t) - 2\cos(t))_0^{\pi} = 2\pi + 4.$$

6. Convert the given points or fuctions from polar to rectangular (Cartesian).

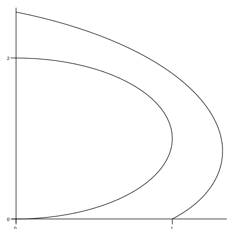
(a) $(2, \pi) = \underline{(-2, 0)}$ $\quad$ $\left(3, \frac{2\pi}{3}\right) = \underline{\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)}$

(b) $\left(4, -\frac{\pi}{6}\right) = \underline{(2\sqrt{3}, -2)}$ $\quad$ $\left(-2, \frac{3\pi}{4}\right) = \underline{(\sqrt{2}, -\sqrt{2})}$

(c) $r = 4$ $\quad \underline{x^2 + y^2 = 16}$

(d) $r = 3\cos(\theta) + 3\sin(\theta)$ (multiply both sides by r) $\underline{\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{2}}$

7. Determine the area between the polar curves $r(\theta) = 1 + \theta$ and $r(\theta) = 2\sin(\theta)$ in the first quadrant.



Area$= \int_0^{\pi/2} \left(\frac{1}{2} + \theta + \frac{1}{2}\theta^2 - 2\sin^2(\theta)\right) d\theta$

$$= \int_0^{\pi/2} \left(\frac{1}{2} + \theta + \frac{1}{2}\theta^2 - 1 + \cos(2\theta)\right) d\theta$$
$$= -\frac{\theta}{2} + \frac{1}{2}\theta^2 + \frac{1}{6}\theta^3 + \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2} = -\frac{\pi}{4} + \frac{\pi^2}{8} + \frac{\pi^3}{48}$$

8. Determine the unit tangent vector to $r(\theta) = 2\sin\theta$ at $\theta = \frac{\pi}{6}$ and add it to the picture above. Note: $x(\theta) = r\cos\theta$ and $y(\theta) = r\sin\theta$.

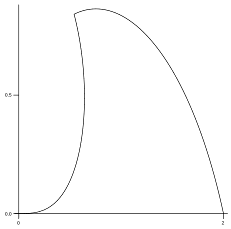
   $x(\theta) = 2\sin\theta\cos\theta = \sin(2\theta)$ and $\frac{dx}{d\theta} = 2\cos(2\theta)$.

   $y(\theta) = 2\sin\theta\sin\theta = 2\sin^2(\theta)$ and $\frac{dy}{d\theta} = 4\sin(\theta)\cos(\theta)$.

   Therefore $\frac{dx}{d\theta}|_{\theta=\pi/6} = 1.$ and $\frac{dy}{d\theta}|_{\theta=\pi/6} = \sqrt{3}$.

   $\vec{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

9. Determine the area between the polar curves $r(t) = \sqrt{t}$ and $r(t) = 2 - t$ as shown.

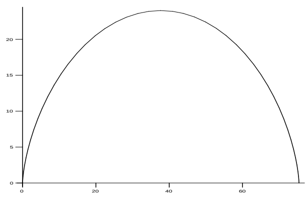
   

   Area$= \int_0^1 \left(2 - 2t + \frac{1}{2}t^2 - \frac{1}{2}t\right)\, dt$

   $= -\frac{5}{4}t^2 + 2t + \frac{1}{6}t^3\Big|_0^1 = \frac{11}{12}$

10. When a bicycle wheel with radius 12 inches turns, the path that is taken by a spot on the tire is called a cycloid and its parametric equations are given as:

$$\begin{aligned} x(t) &= 12t - 12\sin t \\ y(t) &= 12 - 12\cos t \end{aligned}$$

   Determine the arc length of one arch of the cycloid.

   $\left(\frac{dx}{dt}\right)^2 = (12 - 12\cos(t))^2 = 144 - 288\cos(t) + 144\cos^2(t)$

   $\left(\frac{dy}{dt}\right)^2 = 144\sin^2(t)$

Arc length= $\int_0^{2\pi} \sqrt{288 - 288\cos(t)}\, dt = \int_0^{2\pi} 12\sqrt{2(1-\cos(t))}\, dt$

using the half-angle formula

$= \int_0^{2\pi} 12\sqrt{4\sin^2(t/2)}\, dt = \int_0^{2\pi} 24\sin(t/2)\, dt = -48\cos(t/2)\big|_0^{2\pi} = 96$