

Nonhomogeneous Second-Order Differential Equations

To solve $ay'' + by' + cy = f(x)$ we first consider the solution of the form $y = y_c + y_p$ where y_c solves the differential equation $ay'' + by' + cy = 0$ and y_p solves the differential equation $ay'' + by' + cy = f(x)$.

Since the derivative of the sum equals the sum of the derivatives, we will have a final solution of $0 + f(x)$ which gives $f(x)$.

Therefore, we follow a pattern for y_p to yield $f(x)$ when each of y , y' and y'' are substituted into the above equation.

1. Case I: If $f(x)$ is of the form $f(x) = P(x)e^{kx}$ then we set:
 $y_p = Q(x)e^{kx}$ where $Q(x)$ is the general polynomial of the same degree as $P(x)$.
For example if the differential equation is set equal to:
 - (a) $f(x) = x^2e^{3x}$. Set $y_p = (Ax^2 + Bx + C)e^{3x}$.
 - (b) $f(x) = x^3 - 4x + 1$. Set $y_p = Ax^3 + Bx^2 + Cx + D$.
 - (c) $f(x) = 6e^{-2x}$. Set $y_p = Ae^{-2x}$.

Then we substitute y_p , y'_p , and y''_p and solve for the coefficients.

2. Case II: If $f(x)$ is of the form $f(x) = P(x)e^{kx} \sin(mx)$ or $f(x) = P(x)e^{kx} \cos(mx)$, then we set:
 $y_p = Q(x)e^{kx} \cos(mx) + R(x)e^{kx} \sin(mx)$ where $Q(x)$ and $R(x)$ are both general polynomials of the same degree as $P(x)$.
For example if the differential equation is set equal to:
 - (a) $f(x) = 2 \cos(3x)$. Set $y_p = A \cos(3x) + B \sin(3x)$
 - (b) $f(x) = x \cos(x)$. Set $y_p = (Ax + B) \cos(x) + (Cx + D) \sin(x)$
 - (c) $f(x) = e^x \sin(2x)$. Set $y_p = Ae^x \sin(2x) + Be^x \cos(2x)$

If $f(x)$ is a sum of terms, like $f(x) = x^2 + e^{-x} + \cos(x)$, do it as separate problems solving for

$$y_{p1} = Ax^2 + Bx + C,$$

$$y_{p2} = Ae^{-x} \text{ and}$$

$$y_{p3} = A \cos(x) + B \sin(x).$$

Examples

1. $y'' + 2y' + y = 3 \cos(2x)$, $y(0) = 0$, $y'(0) = 1$

$$y = y_c + y_p$$

$r^2 + 2r + 1 = 0$. $r = -1$. Therefore $y_c = Me^{-x} + Nxe^{-x}$ (hold until end).

Now to determine y_p , consider the form:

$$y_p = A \cos(2x) + B \sin(2x). \text{ Then differentiating gives}$$

$$y_p' = -2A \sin(2x) + 2B \cos(2x)$$

$$y_p'' = -4A \cos(2x) - 4B \sin(2x).$$

Substituting gives:

$$-4A \cos(2x) - 4B \sin(2x) - 4A \sin(2x) + 4B \cos(2x) + A \cos(2x) + B \sin(2x) = 3 \cos(2x)$$

$$\text{So that: } -4A + 4B + A = 3 \text{ and } -4B - 4A + B = 0$$

$$\text{The system of equations gives } A = -\frac{9}{25} \text{ and } B = \frac{12}{25}.$$

NOT DONE: We now have

$$y = Me^{-x} + Nxe^{-x} - \frac{9}{25} \cos(2x) + \frac{12}{25} \sin(2x)$$

Substituting the initial conditions gives $M = \frac{9}{25}$ and $N = \frac{10}{25}$. DONE

$$y = \frac{9}{25}e^{-x} + \frac{10}{25}xe^{-x} - \frac{9}{25} \cos(2x) + \frac{12}{25} \sin(2x)$$

2. $y'' + 4y = x^2e^{-x}$, $y(0) = 0$, $y'(0) = 1$

$$y = y_c + y_p$$

$r^2 + 4 = 0$. $r = \pm 2i$. Therefore $y_c = M \cos(2x) + N \sin(2x)$ (hold until end).

Now to determine y_p , consider the form:

$y_p = (Ax^2 + Bx + C)e^{-x}$. Then differentiating gives

$$y_p' = (2Ax + B)e^{-x} - (Ax^2 + Bx + C)e^{-x}$$

$$y_p'' = 2Ae^{-x} - (2Ax + B)e^{-x} - (2Ax + B)e^{-x} + (Ax^2 + Bx + C)e^{-x}.$$

Substituting gives:

$$2Ae^{-x} - 4Axe^{-x} - 2Be^{-x} + Ax^2e^{-x} + Bxe^{-x} + Ce^{-x} + 4Ax^2e^{-x} + 4Bxe^{-x} + 4Ce^{-x} = x^2e^{-x}$$

Gather and group quadratic, linear and constant terms to have: $5A = 1$, which gives $A = \frac{1}{5}$.

$$-4A + B + 4B = 0 \Rightarrow -\frac{4}{5} + 5B = 0 \Rightarrow B = \frac{4}{25}.$$

$$2A - 2B + C + 4C = 0 \Rightarrow \frac{2}{5} - \frac{8}{25} + 5C = 0 \Rightarrow C = -\frac{2}{125}$$

NOT DONE: We now have:

$$y = M \cos(2x) + N \sin(2x) + \left(\frac{1}{5}x^2 + \frac{4}{5}x - \frac{2}{125}\right)e^{-x}$$

Substituting the initial conditions gives $M = \frac{2}{125}$ and $N = \frac{27}{250}$. DONE !

$$y = \frac{2}{125} \cos(2x) + \frac{27}{250} \sin(2x) + \left(\frac{1}{5}x^2 + \frac{4}{5}x - \frac{2}{125}\right)e^{-x}$$