1. Right Triangle Trigonometry

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\cos\theta}{\sin\theta}$$

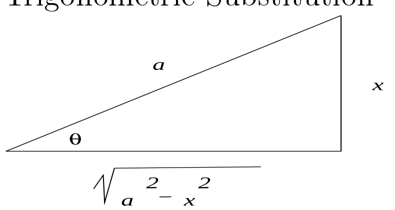
2. Trigonometric Equalities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\tan^2\theta + 1 = \sec^2\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

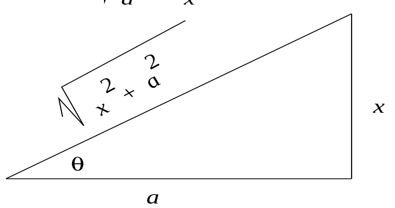
3. More Trigonometric Equalities

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta) \qquad \sin A\cos B = \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta) \qquad \cos A\cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin A\sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

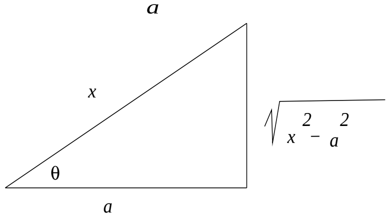
4. Trigonometric Substitution



$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta$



$\sqrt{x^2+a^2}$ $\quad x = a\tan\theta$



$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$

5. Decomposition Examples-Partial Fractions

$$\frac{x-1}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$\frac{3x+5}{(x+7)^2} = \frac{A}{x+7} + \frac{B}{(x+7)^2}$$

$$\frac{7x}{x^3-1} = \frac{7x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\frac{5x^2+2}{(x+1)(x-3)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\frac{x+2}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

6. Exercises with Integration

(a) $\int \sin^2 x \cos^3 x \, dx$.
Save a factor of $\cos x \, dx$ for du and use the identity $\cos^2 x = 1 - \sin^2 x$.
$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx = \int (\sin^2 x - \sin^4 x) \cos x \, dx =$
$\int (u^2 - u^4) \, du$. Letting $u = \sin x$ and $du = \cos x \, dx$.
$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

(b) $\int \frac{\sqrt{x-2}}{x+2} \, dx$. Rationalize.
Let $u = \sqrt{x-2}$ so that $u^2 = x-2$ and $2u \, du = dx$ and $u^2 + 4 = x + 2$.
Then $\int \frac{\sqrt{x-2}}{x+2} \, du = \int \frac{u}{u+4} 2u \, du = \int \frac{2u^2}{u^2+4} \, du = \int (2 - \frac{8}{u^2+4}) \, du$
Make the second term look like the derivative of the arctan function:
$= \int (2 - \frac{2}{1+(u/2)^2}) \, du = 2u - 4 \arctan(u/2) + c$
$\int \frac{\sqrt{x-2}}{x+2} \, dx = 2\sqrt{x-2} - 4 \arctan \frac{\sqrt{x-2}}{2} + c.$

(c) $\int \ln(1+x^2) \, dx$. By parts.

$u = \ln(1+x^2)$	$dV = dx$
$du = \frac{2x}{1+x^2} \, dx$	$V = x$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{x^2}{1+x^2}\,dx = x\ln(1+x^2) - \int (1 - \frac{1}{1+x^2})\,dx$$
$$\int \ln(1+x^2)\,dx == x\ln(1+x^2) - x + \arctan x + c.$$

(d) $\int \cos(\sqrt{x})\,dx$. Rationalize and Substitute.
Let $w = \sqrt{x}$,then $w^2 = x$, and $2w\,dw = dx$. So we have:
$\int 2w \cos w\,dw$. By parts this is: $\begin{array}{l|l} u = 2w & dV = \cos w\,dw \\ du = 2\,dw & V = \sin w \end{array}$
$$\int \cos(\sqrt{x})\,dx = 2w\sin w - \int 2\sin 2w\,dw = 2w\sin w + 2\cos w.$$
$$\int \cos(\sqrt{x})\,dx = 2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x})$$

(e) $\int \frac{x}{x^2+3x+2}\,dx$. Factor and decompose.
$$\frac{x}{(x+2)(x+1)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{-1}{x+1} + \frac{2}{x+2}$$
Note: $A(x+2)+B(x+1) = x$. If $x = -2 \Rightarrow -B = -2$. If $x = -1 \Rightarrow A = -1$.
$$\int \frac{x}{x^2+3x+2}\,dx = \int \left(\frac{-1}{x+1} + \frac{2}{x+2}\right)\,dx = -\ln(x+1) + 2\ln(x+2) + c.$$

(f) $\int \frac{dx}{x\sqrt{9x^2+12x+5}}$. Trig Substitution. Complete the square first.
$$\int \frac{dx}{\sqrt{9x^2+12x+5}} = \int \frac{dx}{\sqrt{(3x+2)^2+1}}.$$ Let $3\tan\theta = 3x+2$. Then $\sec^2\theta\,d\theta = dx$
$$\int \frac{dx}{\sqrt{9x^2+12x+5}} = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + c.$$
Now substitute back $\sec\theta = \sqrt{9x^2+12x+5}$ and $\tan\theta = 3x+2$.
$$\int \frac{dx}{\sqrt{9x^2+12x+5}} = \ln|\sqrt{9x^2+12x+5} + 3x + 2| + c.$$

(g) $\int \frac{e^{2x}}{1+e^x}\,dx$. Let $u = e^x$ Then $du = e^x\,dx$ Now substitute using $e^{2x} = e^x \cdot e^x$:
$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u}{1+u} u\,du = \int \frac{u^2}{1+u}\,du = \int (u - 1 + \frac{1}{1+u})\,du$$
$$= \frac{1}{2}u^2 - u + \ln(1+u) = \frac{1}{2}e^{2x} - e^x + \ln(1+e^x) + c.$$

(h) $\int \frac{x^4}{x^{10}+16}\,dx$. Let $u = x^5$ Then $du = \frac{1}{5}x^4$

$$\int \frac{x^4}{x^{10}+16}\,dx = \frac{1}{5}\int \frac{du}{u^2+16} = \frac{1}{80}\int \frac{du}{(u/4)^2+1}$$

$$\int \frac{x^4}{x^{10}+16} = \frac{1}{20}\arctan(u/4) + c = \frac{1}{20}\arctan\left(\frac{x^4}{4}\right) + c.$$

(i) $$\int \cos(3x)\sin(5x)\,dx = \frac{1}{2}\int (\sin(8x)+\sin(2x))\,dx = \frac{-1}{16}\cos 8x + \frac{-1}{4}\cos 2x + c.$$