Differential Equations Review

1. $y' + 2y = x + e^{-3x} \qquad y(0) = 2$
   $I.F. = e^{\int 2\,dx} = e^{2x}$
   $e^{2x}y' + 2e^{2x}y = xe^{2x} + e^{-x}$
   $e^{2x}y = \int \left(xe^{2x} + e^{-x}\right) dx$
   $e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} - e^{-x} + c \; y = \frac{1}{2}x - \frac{1}{4} - e^{-3x} + \frac{13}{4}e^{-2x}$

2. $x\frac{dy}{dx} + y = 2x\sin x$
   $\frac{dy}{dx} + \frac{1}{x}y = 2\sin x$
   $I.F. = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$
   $xy' + y = 2x\sin x$
   $xy = \int 2x\sin x\,dx$
   $xy = -2x\cos x + \frac{2\sin x}{x} + \frac{c}{x}$

3. $(1 + x^2)y' + y = 3 \qquad y(0) = 2$
   $y' + \frac{1}{1+x^2}y = \frac{3}{1+x^2}$
   $I.F. = e^{\int \frac{1}{1+x^2}\,dx} = e^{\arctan x}$
   $e^{\arctan x}y' + \frac{e^{\arctan x}}{1+x^2}y = \frac{3e^{\arctan x}}{1+x^2}$
   $e^{\arctan x}y = \int \frac{3e^{\arctan x}}{1+x^2}\,dx$
   $e^{\arctan x}y = 3e^{\arctan x} + c$
   $y = 3 - 1e^{-\arctan x}$

4. $y'' + 4y' + 4y = 4\cos(2x) \qquad y(0) = 0 \qquad y'(0) = 1$
   for y_c: $r^2 + 4r + 4 = 0 \quad r = -2 \qquad y_c = Me^{-2x} + Nxe^{-2x}$
   for y_p : $\qquad y_p = A\cos(2x) + B\sin(2x)$
   $y_p' = -2A\sin(2x) + 2B\cos(2x)$ and $y_p'' = -4A\cos(2x) - 4B\sin(2x)$
   Therefore to determine A and B, substitute:

   $\underline{-4A\cos(2x)} - 4B\sin(2x) - 8A\sin(2x) + \underline{8B\sin(2x)} - \underline{4A\cos(2x)} + 4B\sin(2x) = \underline{4\cos(2x)}$

and we have $8B = 4$(from underlined) and $-8A = 0$ (from nonunderlined)
So $y = Me^{-2x} + Nxe^{-2x} + \frac{1}{2}\sin(2x)$
To determine M and N, use initial conditions:
$0 = M$ and $y' = Ne^{-2x} - 2Nxe^{-2x} + \cos x$ So $1 = N + 1$ and $N = 0$.
Therefore: $\qquad y = \frac{1}{2}\sin(2x)$

5. $y" + 9y = xe^{2x}$
for y_c: $r^2 + 9 = 0 \quad r = 0 \pm 3i \qquad y_c = Mcos(3x) + N\sin(3x)$
for y_p: $\quad y_p = (Ax + B)e^{2x}$
$y_p' = Ae^{2x} + 2(Ax + B)e^{2x}$ and $y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4(Ax + B)e^{2x}$
To determine A and B, substitute:
$2Ae^{2x} + 2Ae^{2x} + 4(Ax + B)e^{2x} + 9(Ax + B)e^{2x} = xe^{2x}$
$2A + 2A + \underline{4Ax} + 4B + \underline{9Ax} + 9B = \underline{x}$
$13A = 1$(from underlined) so $A = \frac{1}{13}$
$4A + 13B = 0$ (from nonunderlined) so $B = -\frac{4}{169}$
Therefore:
$$y = M\cos(3x) + N\sin(3x) + \left(\frac{1}{13}x - \frac{4}{169}\right)e^{2x}$$

6. $2y" + 7y' + 3y = 3x^2 + 4 \qquad y(0) = 0 \qquad y'(0) = 1$
for y_c: $2r^2 + 7r + 3 = 0 \quad r = -\frac{1}{2}$ or $r = -3, \qquad y_c = Me^{-x/2} + Ne^{-3x}$
for y_p: $\quad y_p = Ax^2 + Bx + C$ and $y_p' = 2Ax + B$ and $y_p'' = 2A$
To determine A, B and C, substitute:
$4A + \underline{14Ax} + 7B + \underline{\underline{3Ax^2}} + \underline{3Bx} + 3C = \underline{\underline{3x^2}} + 4$
$3A = 3$ (from double-underlined) so $A = 1$.
$14A + 3B = 0$ (from single-underlined) so $B = -\frac{14}{3}$
$4A + 7B + 3C = 4$ (from non-underlined) so $C = -\frac{98}{9}$.
Therefore: $y = Me^{-x/2} + Ne^{-3x} + x^2 - \frac{14}{3}x + \frac{98}{9}$.
To determine M and N, use initial conditions and get:

$$\begin{aligned} M + N &= -\frac{98}{9} \\ -M - 6N &= \frac{34}{3} \end{aligned}$$

$M = -\dfrac{54}{5}$ and $N = \dfrac{4}{9}$
Therefore:

$$y = -\frac{54}{5}e^{-x/2} - \frac{4}{45}e^{-3x} + x^2 - \frac{14}{3}x + \frac{98}{9}$$