

1. Suppose people start smoking in a room of volume $60 m^3$, thereby introducing air containing 5% ($= \frac{1}{20}$) carbon monoxide at a rate of $0.002m^3/\text{min}$ into the room. Assume that the smoky air mixes immediately with the rest of the air, and that the mixture leaves the room at the same rate as it enters.

- (a) Write a differential equation for $y(t)$, the amount of carbon monoxide in the room at time t , in minutes.

This is typical of a mixing problem.

Let $y(t)$ = the amount of carbon in the room at time t .

Then $\frac{dy}{dt}$ = the change in the amount of carbon over time.

The change in carbon over time is the rate carbon comes in per time period minus the rate carbon goes out per time period.

$$\begin{aligned}\frac{dy}{dt} &= \text{rate-in} - \text{rate-out} \\ \frac{dy}{dt} &= \frac{1}{10000} - \frac{2}{1000} \cdot \frac{y}{60} \\ \frac{dy}{dt} &= -\frac{1}{30000}(y - 3)\end{aligned}$$

- (b) Solve the differential equation, assuming there was no carbon monoxide in the room initially.

$$\begin{aligned}\frac{1}{y-3} dy &= -\frac{1}{30000} dt \\ \ln |y-3| &= -\frac{1}{30000} t + c \\ y-3 &= Ae^{-t/30000} \\ y &= 3 - 3e^{-t/30000}\end{aligned}$$

2. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.

- (a) Write a differential equation that is satisfied by y .

If y is the fraction that has heard the rumor, then $1 - y$ is the fraction who have not heard the rumor. Therefore,

$$\frac{dy}{dt} = ky(1 - y)$$

(b) Solve the differential equation.

$$\begin{aligned}\frac{1}{y(1-y)} dy &= k dt \\ \left[\frac{1}{y} + \frac{1}{1-y} \right] dy &= k dt \\ \ln |y| - \ln |1-y| &= kt + c \\ \ln \left| \frac{1-y}{y} \right| &= -kt + c \\ \frac{1-y}{y} &= Ae^{-kt} \\ y &= \frac{1}{1 + Ae^{-kt}}\end{aligned}$$

(c) A small town has 1000 inhabitants. At 8:00 a.m. 80 people have heard the rumor.

This means that if we let 8:00 be time $t = 0$, $\frac{8}{1000} = \frac{1}{125}$ have heard the rumor.

$$y = \frac{1}{1 + 11.5e^{-kt}}$$

By noon half the town has heard it.

This means that at time $t = 4$, $y = 0.5$. Solve for k .