The Cross Product

1. Determine the equation of the plane passing through the point $P(-1,4,8)$ and parallel to the vectors $\vec{u} = \langle 2,7,-3\rangle$ and $\vec{v} = \langle -1,6,2\rangle$.
   Answer:
   A vector normal to $\vec{u}$ and $\vec{v}$ will be normal (perpendicular) to the plane. Such a vector is the cross product determined by:
   $\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 7 & -3 \\ -1 & 6 & 2 \end{pmatrix} = (14+18)\vec{i} - (4-3)\vec{j} + (12+7)\vec{k} = 32\vec{i} - \vec{j} + 19\vec{k}$
   Now choose an arbitrary $Q(x,y,z)$ point on the plane. Then the vector $\vec{PQ} =< x+1, y-4, z-8 >$ is parallel to the plane and perpendicular to the cross product vector. Thus, their dot product is zero:

   $$\langle x+1, y-4, z-8\rangle \cdot \langle 32,-1,19\rangle = 0 \quad \text{or} \quad 32x - y + 19z = 116$$

2. Determine the equation of the plane passing through the points $P(-2,1,5)$ and $Q(4,9,-3)$ and $R(3,-1,0)$.
   Answer:
   The vectors $\vec{PQ} = \langle 6,8,-8\rangle$ and $\vec{PR} = \langle 5,-2,-5\rangle$ are parallel to the plane. Again, a vector normal to the plane is the cross product determined by:
   $\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 8 & -8 \\ 5 & -2 & -5 \end{pmatrix} = (-40-16)\vec{i} - (-30+40)\vec{j} + (-12-40)\vec{k} = -56\vec{i} - 10\vec{j} - 52\vec{k}$
   Again, choose an arbitrary $T(x,y,z)$ point on the plane. Then the vector $\vec{TR} =< x-3, y+1, z-0 >$ is parallel to the plane and perpendicular to the cross product vector. Thus, their dot product is zero:

   $$\langle x-3, y+1, z-0\rangle \cdot \langle -56,-10,-52\rangle = 0 \quad \text{or} \quad 56x + 10y + 52z = 158$$

3. (a) Determine the area of the parallelogram in the xy plane determined by the vectors $\vec{v} = \langle -2,5\rangle$ and $\vec{w} = \langle 4,9\rangle$.
   Answer: (Calculate the absolute value of the determinant)
   $\left|\det\begin{pmatrix} -2 & 5 \\ 4 & 9 \end{pmatrix}\right| = |-18-20| = 38$

   (b) Determine the area of the parallelogram in the xyz space determined by the vectors $\vec{v} = \langle -1,3,4\rangle$ and $\vec{w} = \langle 2,4,9\rangle$.

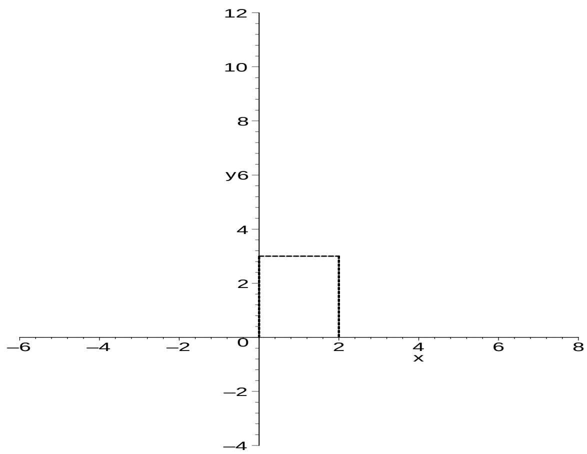
Answer: (Calculate the magnitude of the cross product)

$$\left\| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 4 \\ 2 & 4 & 9 \end{pmatrix} \right\| = ||(27-16)\vec{i} - (-9-8)\vec{j} + (-4-6)\vec{k}|| = \sqrt{393}$$

4. Determine the volume of the parallelopiped formed by the vectors $\vec{a} = \langle 1, 3, -2\rangle$ and $\vec{b} = \langle 3, -2, 5\rangle$ and $\vec{c} = \langle 6, 2, 3\rangle$.
Answer: (Calculate the absolute value of the determinant)

$$\left| \det \begin{pmatrix} 1 & 3 & -2 \\ 3 & -2 & 5 \\ 6 & 2 & 3 \end{pmatrix} \right| = |1(-6-10) - 3(9-30) - 2(6+12)| = 11$$

5. If $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ and S is the 2×3 rectangle in the first quadrant as shown, sketch $L(S)$ indicating $L(\vec{a})$ and $L(\vec{b})$ where $\vec{a} = \langle 2, 0\rangle$ and $\vec{b} = \langle 0, 3\rangle$.



Answer:
$L\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $L\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -9 \\ 12 \end{pmatrix}$.
Draw these vectors on the graph to show the linear transformation of the rectangle to the parallelogram.