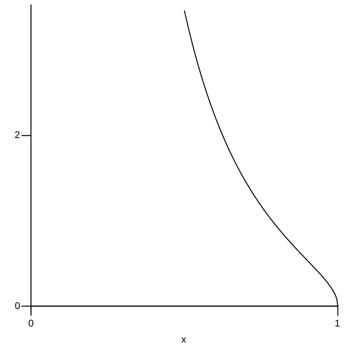
1. Determine the average value of the function $f(x) = x^3 \ln x$ on $1 \leq x \leq 10$.

$$\begin{aligned} A.V. &= \frac{1}{9}\int_1^{10} x^3 \ln x \, dx \\ &= \frac{1}{9}\left(\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right)_1^{10} \\ &= \frac{1}{9}\left(\frac{10000}{4}\ln 10 - \frac{10000}{16} + \frac{1}{16}\right) \end{aligned}$$

2. Determine the area bounded by the x-axis and the curve $y = \dfrac{\sqrt{1-x^2}}{x^2}$ on $\dfrac{1}{2} \leq x \leq 1$.

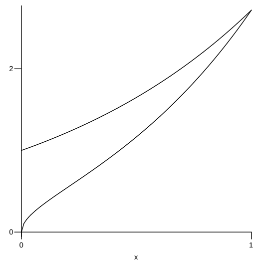


Set up triangle with hypotenuse $c = 1$, opposite side $a = x$ and adjacent side $b = \sqrt{1-x^2}$.
Then $x = \sin\theta$, $dx = \cos\theta \, d\theta$ and $\sqrt{1-x^2} = \cos\theta$

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x^2}\,dx &= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta \, d\theta \\ &= \int \cot^2\theta \, d\theta \\ &= \int (\csc^2\theta - 1)\, d\theta \\ &= -\cot\theta - \theta \\ &= -\frac{\sqrt{1-x^2}}{x} - \arcsin x \end{aligned}$$

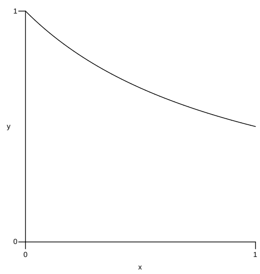
$$\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx = \left(\frac{\sqrt{1-x^2}}{x} - \arcsin x\right)_{1/2}^{1} = -\frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6} = \sqrt{3} - \frac{\pi}{3}$$

3. Determine the volume of the solid generated by rotating about the x-axis the region in the first quadrant bounded by the functions $f(x) = \sqrt{x}\, e^x$ and $y = e^x$ on $0 \leq x \leq 1$.



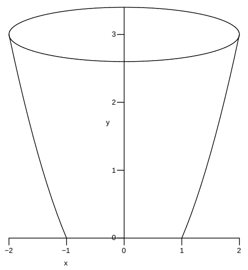
$$\begin{aligned} V &= \pi \int_0^1 \left((e^x)^2 - (\sqrt{x}\, e^x)^2\right) dx \\ &= \pi \int_0^1 \left(e^{2x} - xe^{2x}\right) dx \\ &= \pi \left(\frac{1}{2}e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x}\right)_0^1 \\ &= \pi \left(\frac{1}{4}e^2 - \frac{3}{4}\right) \end{aligned}$$

4. Determine the volume of the solid generated by rotating about the y-axis the region bounded by the y-axis, the x-axis, $x = 1$ and the function $f(x) = \dfrac{3}{x^2 + 5x + 4}$.



$$\begin{aligned} V &= 2\pi \int_0^1 \frac{3x}{x^2 + 5x + 4}\, dx \\ &= 2\pi \int_0^1 \left(\frac{4}{x+4} - \frac{1}{x+1}\right) dx \\ &= 2\pi \left(4\ln(x+4) - \ln(x+1)\right)_0^1 \\ &= 2\pi \left(4\ln(5) - \ln(2) - 4\ln(4)\right) \end{aligned}$$

5. Determine the work to emty a bowl filled with water (62.4 lb/ft^3) out of the top if the bowl fits the rotated parabola $y = x^2 - 1$ for $1 \le x \le 2$ in feet.



$$\begin{aligned} W &= 62.4\pi \int_0^3 \left(\sqrt{y+1}\right)^2 (3-y)\,dy \\ &= 62.4\pi \int_0^1 (-y^2+2y+3)\,dy \\ &= 62.4\pi \left(-\frac{1}{3}y^3+y^2+3y\right)_0^3 = 62.4(9)\pi \end{aligned}$$

6. Evaluate the integral if it converges. If it diverges, show the diverging limit.

(a) $\displaystyle\int_2^\infty \frac{3}{(x-1)^2}\,dx$ Converges

$$\int_2^\infty \frac{3}{(x-1)^2}\,dx = -\frac{3}{x-1}\Bigg|_{x=2}^{x=t\to\infty} = 0-(-3) = 3$$

(b) $\displaystyle\int_1^2 \frac{x}{x^2-1}\,dx$ Diverges

$$\int_1^2 \frac{x}{x^2-1}\,dx = \frac{1}{2}\ln\left(x^2-1\right)\Bigg|_{x\to1}^{x=2} \to \infty \text{ as x} \to 1$$

(c) $\displaystyle\int_1^2 \frac{3x}{\sqrt{x^2-1}}\,dx$

$$\int_1^2 \frac{3x}{\sqrt{x^2-1}}\,dx = 3\sqrt{x^2-1}\Big|_{x\to1}^{2} = 3\sqrt{3}-0 = 3\sqrt{3}$$

7. Vector Fundamentals

(a) Determine the unit vector in the direction of the given vector:

$\vec{w} = 2\vec{\imath} - 5\vec{\jmath} + 8\vec{k}$ $\quad \underline{\vec{u}_{\vec{w}} = \left\langle \frac{2}{\sqrt{93}}, -\frac{5}{\sqrt{93}}, \frac{8}{\sqrt{93}} \right\rangle}$

(b) Determine the scalar projection of $\vec{F} = \langle 4, 7 \rangle$ onto $\vec{d} = \langle 9, 2 \rangle$.

$$\text{comp}_{\vec{d}}\vec{F} = \frac{\vec{F} \cdot \vec{d}}{|\vec{d}|} = \frac{50}{\sqrt{85}}$$

(c) Determine the angle θ (in degrees) between the vectors $\vec{v} = \langle -1, 7, 0 \rangle$ and $\vec{w} = \langle 3, 4, 5 \rangle$.

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{-3 + 28}{\sqrt{50}\sqrt{50}} = \frac{25}{50} = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

(d) Determine the vector equation of the line passing through points $P(-2, 4, 1)$ and $Q(3, 3, 3)$.

Direction vector is $\vec{v} = \langle 3 + 2, 3 - 4, 3 - 1 \rangle = \langle 5, -1, 2 \rangle$
Line: $\vec{r}\,(t) = \langle\, -2 + 5t, 4 - 1t, 1 + 2t\, \rangle$

(e) Determine the volume of the parallelopiped formed by the vectors $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle 5, -1, 4 \rangle$ and $\vec{c} = \langle 1, 3, 8 \rangle$.

$$\left| \det \begin{pmatrix} 2 & -4 & 1 \\ 5 & -1 & 4 \\ 1 & 3 & 8 \end{pmatrix} \right| = |2(-8 - 12) - (-4)(40 - 4) + 1(15 + 1)| = 120$$

8. For what value(s) of a are the vectors $\langle a^2, -1, 3\rangle$ and $\langle 2, a, -5\rangle$ orthogonal (perpendicular)?

   $\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$. Therefore: $2a^2 - a - 15 = 0$
   and $(2a+5)(a-3) = 0 \;\Rightarrow a = 5/3$ or $a = 3$

9. Determine the area of the parallelogram formed by the vectors $\vec{v} = \langle 1, 1, 4\rangle$ and $\vec{w} = \langle -2, 3, 2\rangle$

   $$\vec{n} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ -2 & 3 & 2 \end{pmatrix} = -10\vec{i} - 10\vec{j} + 5\vec{k}$$
   Area$=|\vec{n}| = \sqrt{225} = 15$.

10. Determine the equation of the plane contains the vectors $\vec{v} = \langle 4, 2, -1\rangle$ and $\vec{w} = \langle 1, 3, -3\rangle$ and contains the point $P(2, 3, -5)$.

    $$\vec{n} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & -1 \\ 1 & 3 & -3 \end{pmatrix} = \langle -3, 11, 10\rangle$$
    $\langle -3, 10, 11\rangle \cdot \langle x-2, y-3, z+5\rangle = 0$ implies
    Plane: $-3x + 11y + 10 = -23$