Exam III Name ______________________________

1. Solve the initial-value differential equation

$$(1+x^2)y' + 2xy = 3\sqrt{x} \qquad\qquad y(0) = 2.$$

answer:

$$\begin{aligned} y' + P(x)\, y &= Q(x) \\ y' + \frac{2x}{1+x^2}\, y = \frac{3\sqrt{x}}{1+x^2} \end{aligned}$$

the integrating factor is:

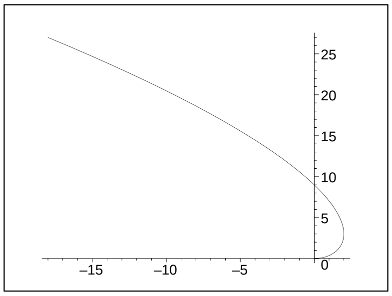
$$\begin{aligned} I(x) &= e^{\int \frac{2x}{1+x^2}} \\ I(x) &= e^{\ln(1+x^2)} = 1+x^2 \end{aligned}$$

so we just get right back to:

$$\begin{aligned} (1+x^2)\, y' + 2x\, y &= 3\sqrt{x} \\ \left((1+x^2)\, y\right)' &= 3\sqrt{x} \\ (1+x^2)\, y &= \int 3\sqrt{x}\, dx \\ (1+x^2)\, y &= 2\sqrt{x} + c \\ y(x) &= \frac{2\sqrt{x}}{1+x^2} + \frac{c}{1+x^2} \\ y(x) &= \frac{2\sqrt{x}}{1+x^2} + \frac{4}{1+x^2} \end{aligned}$$

2. Determine the arclength of the curve given by

$$x(t) = 3t - t^3, \qquad y(t) = 3t^2 \qquad\qquad \text{for } \ 0 \le t \le 3.$$



answer:

$x'(t) = 3 - 3t^2$ $\qquad$ $y'(t) = 6t$

$(x'(t))^2 = 9 - 18t^2 + 9t^4$ $\qquad$ $(y'(t))^2 = 36t^2$

$$\begin{aligned}
\text{Arclength} &= \int_0^3 \sqrt{9 + 18t^2 + 9t^4}\, dt \\
&= \int_0^3 \sqrt{(3 + 3t^2)^2}\, dt \\
&= \int_0^3 (3 + 3t^2)\, dt \\
&= 3t + t^3 \Big|_{t=0}^{t=3} = 36
\end{aligned}$$

3. Set up the equations for $(\overline{x}, \overline{y})$, the center of mass, centroid of the region bounded by $y = \sin(x)$ for $0 \le x \le \pi/2$.

$$\overline{x} = \frac{\int_0^{\pi/2} x\ \sin x dx}{\int_0^{\pi/2}\ \sin x\, dx}$$

$$\overline{y} = \frac{\int_0^{\pi/2}\ \frac{1}{2}\sin^2 x\, dx}{\int_0^{\pi/2}\ \sin x\, dx}$$

4. State yes or no for convergence of the given series.

(a) $\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$ $\underline{\mathit{mboxdiverges}}$

(d) $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ $\underline{\text{converges}}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{(n+1)}$ <u>diverges</u>

(e) $\sum_{n=1}^{\infty} \frac{1}{3n}$ <u>diverges</u>

(c) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n}$ <u>converges</u>

5. Determine the Taylor Series for $f(x) = \frac{1}{\sqrt{4+2x}}$ for $-2 < x < 2$. answer:

$$\begin{aligned} \frac{1}{\sqrt{4+2x}} &= \frac{1}{\sqrt{4(1+\frac{x}{2})}} \\ &= \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1/2} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^n}{2^{n+1}} \end{aligned}$$

6. Determine the Taylor Series for $f(x) = \frac{1}{x}$ about $x = 1$.

answer: $\frac{1}{x} = \sum_{n=0}^{\infty} c_n (x-1)^n = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n$

Determine the constant coefficients by determining the n$^{\text{th}}$ derivative.

$$\begin{aligned} f^0(x) = \frac{1}{x} &\rightarrow \frac{1}{1} = 1 \\ f^1(x) = -\frac{1}{x^2} &\rightarrow frac11 = -1 \\ f^2(x) = \frac{2}{x^3} &\rightarrow \frac{2}{1} = 2 \\ f^3(x) = -\frac{3\cdot 2}{x^4} &\rightarrow -\frac{3\cdot 2}{1} = -3! \\ f^4(x) = \frac{4\cdot 3\cdot 2}{x^5} &\rightarrow \frac{4\cdot 3\cdot 2}{1} = 4! \\ \vdots &\rightarrow \vdots \\ f^n(x) = \frac{(-1)^n n!}{x^{n+1}} &\rightarrow f^n(1) = (-1)^n n! \end{aligned}$$

So that: $\dfrac{1}{x} = \sum_{n=0}^{\infty} \dfrac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

7. Determine the sum of the series.

(a) $\sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n \quad = \underline{\dfrac{1}{1-3/7} = \dfrac{7}{4}}$

(b) $\sum_{n=0}^{\infty} \dfrac{2^n}{n!} \quad = \underline{e^2}$

(c) $\sum_{n=0}^{\infty} (2x)^n \quad = \underline{\dfrac{1}{1-2x}}$ if $\underline{-1/2 < x < 1/2}$