Final Practice:

1. Find the inverse function \( f^{-1}(x) \) of \( f(x) = \frac{x - 2}{3x + 4} \)

Set \( x = \frac{y - 2}{3y + 4} \) and multiply both sides by \( 3y + 4 \).

\[
f^{-1}(x) = -\frac{4x + 2}{3x - 1}
\]

2. Use a linear approximation to estimate \( \sqrt[3]{7.7} \).

Set \( x_0 = 8 \), \( f(x) = \sqrt[3]{x} \) then \( f'(x) = \frac{1}{3x^{2/3}} \) and \( y = 2 + \frac{1}{12}(x - 8) \).

\[
\sqrt[3]{7.7} \approx 2 + \frac{1}{12} \cdot \frac{-3}{10} = \frac{39}{40}
\]

3. Use the Limit Definition of the Derivative to find the derivative of \( f(x) = \sqrt[3]{2x + 3} \) at the point \( a = 3 \). NO CREDIT will be given if Limit Definition is not used.

\[
\lim_{x \to 3} \frac{\sqrt[3]{2x + 3} - 3}{x - 3} = \lim_{x \to 3} \left( \frac{\sqrt[3]{2x + 3} - 3}{x - 3} \right) \left( \frac{\sqrt[3]{2x + 3} + 3}{\sqrt[3]{2x + 3} + 3} \right)
\]

\[
= \lim_{x \to 3} \frac{2(x - 3)}{(x - 3)(\sqrt[3]{2x + 3} + 3)} = \frac{1}{3}
\]

4. Find the slope of the tangent line to the curve below at the point (1, 2)

\( x^4y^2 + 6x^5 - y^3 + 2x = 4 \)

\[
4x^3y^2 + x^42y \frac{dy}{dx} + 30x^4 - 3y^2 \frac{dy}{dx} + 2 = 0.
\]

\[
4(1)(2^2) + 1(2)(2) \frac{dy}{dx} + 30 - 3(2^2) \frac{dy}{dx} + 2 = 0. \quad \frac{dy}{dx} = 6
\]

5. Use Newton’s Method with initial approximation \( x_1 = 0 \) to find the second approximation \( x_2 \) and the third approximation \( x_3 \) to the root of the equation \( x^3 + x^2 + x - 1 = 0 \).

\[
x_2 = 0 - \frac{-1}{1} = 1 \quad x_3 = 1 - \frac{2}{6} = \frac{2}{3}
\]
6. A particle is moving in a straight line and its acceleration at the time \( t \) is given by \( a(t) = \cos t - 6t^2 + 1 \) (m/s\(^2\)). If the initial velocity of the particle is \( v(0) = 2 \) (m/s) and the initial position is \( s(0) = 3 \) (m), find the position of the particle at the time \( t = \pi \) (s).

\[
v(t) = \sin t - 2t^3 + t + 2
\]

\[
s(t) = -\cos t - \frac{1}{2}t^4 + \frac{1}{2}t^2 + 2t + 4 \quad s(\pi) = 5 - \frac{\pi^4}{2} + \frac{\pi^2}{2}.
\]

7. A ladder 3 meters long is leaning against a vertical wall. The base of the ladder starts to slide away from the wall at 5 m/min. How fast is the angle between the ladder and the ground changing when the base is 1 meter away from the wall?

\[
\cos \theta = \frac{x}{3}.
\]

Differentiating with respect to \( t \) gives:

\[
-\sin \theta \frac{d\theta}{dt} = \frac{1}{3} \frac{dx}{dt}
\]

\[
\frac{d\theta}{dt} = -\frac{5}{\sqrt{8}} \quad \text{Speed} = \frac{5}{\sqrt{8}} \text{ rad/min}
\]

8. A box with a rectangular base without a lid must have a volume of 18 ft\(^3\). The length of the base of the box is three times its width. Find the dimensions of the box that minimize the amount of material used.

Minimize \textbf{material} under constraint \textbf{volume=18}.

\[
M = 3x^2 + 8xh
\]

\[
3x^2h = 18
\]

\[
M(x) = 3x^2 + \frac{48}{x}. \text{ Differentiate: } M'(x) = 6x - \frac{48}{x^2}.
\]

\[
x = 2. \quad \text{base} = 2\, \text{ft} \times 6\, \text{ft} \quad \text{height} = \frac{3}{2}\, \text{ft}.
\]

9. Evaluate each of the following limits. All work must be shown.

(a) \[
\lim_{x \to +\infty} \frac{3x^2 + x + 1}{x^3 - 2} = \lim_{x \to +\infty} \frac{\frac{3x^2}{x^3} + \frac{x}{x^3} + \frac{1}{x^3}}{\frac{x^3}{x^3} - \frac{2}{x^3}} = 0
\]

(b) \[
\lim_{x \to 0} \frac{e^{2x} + \ln (3x + 1) - 1}{\sin (3x)} = \lim_{x \to 0} \frac{2e^{2x} + \frac{3}{3x+1}}{3\cos (3x)} = \frac{5}{3}
\]

(c) \[
\lim_{x \to +\infty} x \left( \tan \left( \frac{2}{x} \right) \right) = \lim_{x \to +\infty} \frac{\tan \left( \frac{2}{x} \right)}{\frac{2}{x}} = \lim_{x \to +\infty} \frac{(\sec^2 \left( \frac{2}{x} \right) \left( \frac{2}{x^2} \right)}{x} = 2
\]
\[
\lim_{x \to 0^+} (1 - \sin x)^{3/x} = P = e^{-3}
\]
\[
\ln P = \lim_{x \to 0^+} \frac{3 \ln (1 - \sin x)}{x} = \lim_{x \to 0^+} \frac{-3 \cos x}{1 - \sin x} = -3
\]
\[
\lim_{x \to -5^+} \frac{x^2 - 6x + 5}{|5 - x|} = \lim_{x \to -5^+} \frac{(x - 5)(x - 1)}{|5 - x|} = 4
\]

10. Find the derivative of the following functions. You do not have to simplify your answers.

(a) \( f(x) = \log_2 x + e^{\sin x} + \frac{1}{\sqrt{x}} \)
\[
f'(x) = \frac{1}{x \ln 2} + e^{\sin x} \cos x - \frac{1}{5x^{6/5}}
\]
(b) \( g(x) = (3x - 1) \sin^{-1} (2x) \)
\[
g'(x) = 3 \sin^{-1} (2x) + \frac{2(3x - 1)}{\sqrt{1 - 4x^2}}
\]
(c) \( h(t) = \frac{3^t}{\cos t} \)
\[
h'(t) = \frac{(\cos t)3^t 3t^2 \ln 3 + 3^t \sin t}{\cos^2 t}
\]
(d) \( f(x) = (\tan (5x))^{\sqrt{x}} \)
\[
f'(x) = (\tan (5x))^x \left( \frac{\ln (\tan (5x))}{2 \sqrt{x}} + \sqrt{x} \frac{5 \sec^2 (5x)}{\tan (5x)} \right)
\]
(e) \( f(x) = \int_{3}^{x} 2^{-t^2} \, dt \)
\[
f'(x) = 2^{-x^2}
\]

11. Evaluate the following integrals

(a) \( \int_{1}^{2} \frac{(x + 1)(x + 2)}{x} \, dx = \int_{1}^{2} \left( x + 3 + \frac{2}{x} \right) = \frac{1}{2} x^2 + 3x + 2 \ln x \bigg|_{x=2}^{x=1} = 4.5 + 2 \ln 2 \)
(b) \( \int_{0}^{1} \frac{1}{t^2 + 1} \, dt = \arctan (t) \bigg|_{t=0}^{t=1} = \frac{\pi}{4} \)
(c) \( \int \sin^2 t \cos^3 t \, dt = \int \sin^2 t (1 - \sin^2 t) \cos t \, dt = \int (\sin^2 t - \sin^4 t) \cos t \, dt \)
\[
= \frac{1}{3} \sin^3 t - \frac{1}{5} \sin^5 t + c
\]
(d) $\int_1^4 \frac{5\sqrt{x}}{\sqrt{x}} \, dx = \frac{2}{\ln 5} (5\sqrt{x})\bigg|_1^4 = \frac{40}{\ln 5}$

(e) $\int x^3 \ln (3x) \, dx = \frac{1}{4} x^4 \ln (3x) - \int \frac{1}{4} x^3 \, dx = \frac{1}{4} \ln (3x) - \frac{1}{16} x^4 + c$

(f) $\int \frac{t}{t^2 + 1} \, dt = \frac{1}{2} \ln (t^2 + 1) + c$

12. Find the absolute maximum and the absolute minimum values of $f(x) = \frac{x^3}{3} - 3x^2 + 5$ on the interval $[-3, 1]$.

$f'(x) = x^2 - 6x$ and is equal to zero when $x = 0, 6$ but only $x = 0$ in $[-3, 1]$.

$f(-3) = -31$, the minimum. $f(0) = 5$, the maximum. $f(1) = \frac{2}{3}$.

13. Given the function

$$f(x) = \frac{(x - 1)^2}{(x + 2)(x - 4)}$$

and its derivatives:

$$f'(x) = \frac{18(1 - x)}{(x + 2)^2(x - 4)^2}, \quad f''(x) = \frac{54(x^2 - 2x + 4)}{(x + 2)^3(x - 4)^3}$$

(a) Find all horizontal and vertical asymptotes, if any.

Vertical asymptote at $x = -2$ and $x = 4$. Horizontal asymptote at $y = 1$.

(b) Determine on what intervals $f(x)$ is increasing or decreasing, and find all local maximum and minimum values.

$x < -2, f'(x) > 0, f(x)$ is increasing.

$-2 < x < 1, f'(x) > 0, f(x)$ is increasing.

$1 < x < 4, f'(x) < 0, f(x)$ is decreasing.

$x > 4, f'(x) < 0, f(x)$ is decreasing.

Local maximum at $x = 1$.

(c) Determine on what intervals $f(x)$ is concave up and concave down, and find all inflection points, if any.

No inflection points since numerator is never zero.

$x < -2$, concave up. $-2 < x < 4$, concave down. $x > 4$, concave up.

(d) Use a coordinate axes to sketch the graph of $f(x)$. 