Solitary waves in FPU lattices with alternating bond potentials

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We consider an infinite chain of alternating stiffer and softer bonds with constant ratio of elastic moduli. Using multiple scale analysis in the limit of high mismatch between the moduli, we derive the asymptotic orthogonality condition for the ratio of the moduli at which the system supports genuine solitary waves. We show that under some generic conditions, there exists an infinite sequence of such ratios accumulating to zero. This result is illustrated by an exact calculation for a Toda lattice with alternating bonds and corroborated by numerical simulations.

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1. Introduction

Solitary waves are ubiquitous in many settings, from photonic crystals and electrical networks to granular metamaterials and polymer chains [1–5]. Arising as a result of an interplay between dispersion and nonlinearity, these spatially localized traveling waves transport energy through the system. Following their discovery [6] in the Fermi–Pasta–Ulam (FPU) system [7], there has been an enormous amount of literature on solitary waves in lattices and other dispersive nonlinear systems [8–10].

While most of this work has been devoted to solitary waves in homogeneous media, existence of such excitations in periodic heterogeneous lattices, which arise in modeling proteins and crystalline structures, has also been a subject of much interest [11–22]. Numerical simulations [15–17,20,22] have shown that for generic parameter values, these systems do not support genuine solitary waves: instead, a propagating pulse radiates energy in its wake in the form of oscillations associated with the optical branch of the dispersion relation. However, recent studies [20–22] of diatomic FPU lattices with alternating masses but homogeneous interaction potentials have identified an orthogonality condition in the asymptotic limit of high mismatch between the two masses that yields a sequence of mass ratios for which the amplitude of the oscillations vanishes up to the order of the approximation. These findings, corroborated by numerical simulations in this and earlier work [17], as well as experimental results in diatomic granular crystals [23], strongly suggest existence of genuine solitary waves at certain isolated antiresonance values of the mass ratio. The same phenomenon has been observed in locally resonant granular chains [24]. Such waves in fact belong to the general class of embedded solitons [25–28], nonlinear solitary waves that exist in resonance with dispersive linear waves at isolated parameter values.

In this work we consider a different type of periodic heterogeneous lattice, with uniform masses but alternating stiffer and softer bonds. In the asymptotic limit of high mismatch between the elastic moduli, the system has two well separated time scales, the slow scale governing the dynamics of the centers of mass of the stiffer bonds and the fast time scale on which the centers of mass of the softer bonds oscillate. Using multiscale analysis, we derive the orthogonality condition, which under some generic assumptions yields an infinite sequence of the modulus ratios that approximate antiresonance values supporting the formation of genuine solitary waves with no oscillations in their wake. As in [22], we then consider the Toda potential [29] for which the calculation becomes explicit, and the results are confirmed by numerical simulations.

Our analysis focuses on potentials with nonzero elastic moduli, which yield a characteristic linear oscillation frequency. The case of a “sonic vacuum”, where such frequency is zero, requires a somewhat different scaling in the asymptotic analysis. Such example, a granular chain with alternating Hertzian potentials, was recently considered in [30], where an asymptotic approach was used to generate a sequence of contact stiffness ratios yielding solitary waves, and the results were supported by numerical simulations.

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and experiments on a chain of cylinders stacked at suitably tuned alternating contact angles.

The paper is organized as follows. In Section 2 we formulate the problem. Asymptotic analysis is performed in Section 3. In Section 4 we specialize the results to the case of Toda potential. We conclude with Section 5, where we discuss the results, draw our main conclusions and identify future directions.

2. Problem formulation

Consider an infinite chain of identical masses with interaction potentials alternating between \( \phi(r) \) and \((1/\varepsilon)\phi(r)\). Here the parameter \( \varepsilon \), \( 0 < \varepsilon < 1 \), measures the mismatch between the elastic moduli of the bonds, and we assume that the potential \( \phi(r) \) is sufficiently smooth and convex: \( \phi \in C^2, \phi'(r) > 0 \). We can always rescale the potential so that \( \phi'(0) = 1 \) and \( \phi(0) = \phi'(0) = 0 \). We assume that even-numbered masses are connected to their nearest neighbors on the left by the stiffer bonds and the nearest neighbors on the right by the softer ones. The rescaled equations of motion are then, for any integer \( p \),

\[
\begin{align*}
q_{2p-1}^{(t)} &= \frac{1}{\varepsilon} \phi(q_{2p} - q_{2p-1}) - \phi(q_{2p-1} - q_{2p-2}) \\
q_{2p}^{(t)} &= \phi(q_{2p+1} - q_{2p}) - \frac{1}{\varepsilon} \phi(q_{2p} - q_{2p-1}),
\end{align*}
\]

where \( q_{tm}(t) \) and \( q_m(t) \) denote the displacement and acceleration, respectively, of \( m \)th mass from equilibrium at time \( t \). Multiplying the equations by \( \varepsilon \), we obtain

\[
\begin{align*}
\varepsilon q_{2p-1}^{(t)} &= \phi(q_{2p} - q_{2p-1}) - \varepsilon \phi(q_{2p-1} - q_{2p-2}) \\
\varepsilon q_{2p}^{(t)} &= \varepsilon \phi(q_{2p+1} - q_{2p}) - \phi(q_{2p} - q_{2p-1}).
\end{align*}
\]

Note that the limiting case \( \varepsilon = 0 \), when the stiffer bonds (connecting \( 2p \)th and \( 2p-1 \)th masses) become rigid, become rigid, and still become rigid, which in view of \( \phi'(0) = 0 \) and our assumption of monotonicity of \( \phi(r) \) implies, not surprisingly, that \( q_{2p} = q_{2p-1} \) in this limit. Note also that adding the equations in (1) we obtain the equation governing the dynamics of the center of mass of the stiffer bond:

\[
q_{2p-1} + q_{2p} = \phi(q_{2p+1} - q_{2p}) - \phi(q_{2p-1} - q_{2p-2}).
\]

In the rigid limit \( \varepsilon = 0 \) we have \( q_{2p} = q_{2p-1} \), so this yields

\[
2q_{2p} = \phi(q_{2p+1} - q_{2p}) - \phi(q_{2p-1} - q_{2p-2}).
\]

This equation describes the FPU lattice of double masses (pairs of masses connected by rigid bonds). Clearly, we also have FPU dynamics (for single masses) in another limiting case when the potentials coincide, i.e., \( \varepsilon = 1 \). Based on the existence result proved in [31], we expect formation of solitary waves with supersonic velocities in these two limiting cases under some fairly general assumptions on \( \phi(r) \).

For generic \( \varepsilon \) values away from these two limits numerical simulations show that velocity pulses propagating through the lattice have oscillatory tails; see Fig. 1 for an example. To see the origin of these oscillations, we linearize equations (1) about the zero-strain state and seek solutions of the linearized problem in the plane-wave form \( q_{2p+1} = A \exp[i(\omega t - pk)], q_{2p} = B \exp[i(\omega t - pk)] \).

This yields the dispersion relation

\[
\omega = \omega_\varepsilon(k) = \sqrt{1 + \frac{1}{\varepsilon k^2} + \frac{2}{\varepsilon} \cos k}
\]

between the frequency \( \omega \) and the wave number \( k \) that has two branches: acoustic, \( \omega_\varepsilon(k) \), and optical, \( \omega_\varepsilon(k) \). As illustrated in Fig. 2, in the supersonic regime one thus generically expects resonant interaction of a propagating solitary pulse with the linear waves associated with the optical branch of the dispersion relation. As a result of this resonant interaction, for generic \( \varepsilon \) values such pulses emit small-amplitude oscillations in their wake and slightly decrease in amplitude as they propagate along the lattice.

We seek a condition for special antiresonance \( \varepsilon \) values at which the oscillations disappear, and a genuine solitary wave propagates through the lattice, in the sense that the strains in the stiffer and softer bonds are localized traveling waves: \( q_{2p} - q_{2p-1} = R(\xi) \), \( q_{2p-1} - q_{2p-2} = W(\xi) \), where \( \xi = p - ct \) is the traveling wave coordinate, and \( R(\xi), W(\xi) \to 0 \) as \( |\xi| \to \infty \). As we show in the following section, this condition can be obtained in the asymptotic limit \( 0 < \varepsilon \ll 1 \) when time scales associated with slow dynamics of the centers of mass of the stiffer bonds and fast oscillations of the centers of mass of the softer ones are well separated.

3. Asymptotic analysis

We now consider the asymptotic limit \( 0 < \varepsilon \ll 1 \) and use the method of multiple scales, seeking solutions of (2) in the form

\[
q_m = x_m(t) + \varepsilon y_m(t) + O(\varepsilon^2), \quad t = t/\sqrt{\varepsilon},
\]

where \( \tau \) is the fast time. Observe, however, that the dynamics of the centers of mass of the stiffer and softer bonds is governed by (3) and

\[
\varepsilon(q_{2p+1} + q_{2p}) = \phi(q_{2p+2} - q_{2p+1}) - \phi(q_{2p} - q_{2p-1}).
\]

respectively. This suggests that the fast oscillations of the centers of mass on the stiffer bonds (described by (3)) have amplitude that is an order of \( \varepsilon \) smaller than the amplitude of the corresponding oscillations for the softer bonds (governed by (6)), as confirmed by

![Fig. 1. Velocity profiles \( q_m(t) \) at \( m = 201 \) (grey curve) and \( m = 202 \) (black curve) in a lattice with Toda bond potentials alternating between \( \phi(r) \) given by (24) and \( \phi(r)/\varepsilon \). Here \( \varepsilon = 0.07 \), and the initial conditions are determined from (16), (25), with \( k = 2 \).](image1)

![Fig. 2. Phase velocity as a function of the wave number in the linearized problem for the acoustic (solid bold) and optical (dashed bold) branches. Here \( \varepsilon = 0.07 \). The thin line separates the supersonic and supersonic velocity regimes.](image2)
numerical simulations. In other words, we have (neglecting higher order terms)

\[ q_{2p-1} + q_{2p} \approx \zeta_{2p}(\tau) + \epsilon^2 \eta_{2p}(\tau) \]

\[ q_{2p} + q_{2p+1} \approx \zeta_{2p+1}(\tau) + \epsilon \eta_{2p+1}(\tau) \]

for some functions \( \zeta_{\nu}(\tau) \) and \( \eta_{\nu}(\tau) \). The first of these implies that in (5) we must require that

\[ y_{2p}(\tau) + y_{2p-1}(\tau) = 0 \]

for any integer \( p \). Substituting (5) into (2), we obtain, expanding the right hand side in Taylor series and neglecting terms of \( O(\epsilon^2) \),

\[ \epsilon x_{2p-1} + \epsilon y_{2p-1} \approx \phi(x_{2p} - x_{2p-1}) \]

\[ + \epsilon \phi'(x_{2p} - x_{2p-1})(y_{2p} - y_{2p-1}) - \epsilon \phi(x_{2p-1} - x_{2p-2}) \]

\[ \epsilon x_{2p} + \epsilon y_{2p} \approx \phi(x_{2p+1} - x_{2p}) - \phi(x_{2p} - x_{2p-1}) \]

\[ - \epsilon \phi'(x_{2p} - x_{2p-1})(y_{2p} - y_{2p-1}) \]

where primes denote derivatives with respect to the fast time \( \tau \). \( O(1) \) terms in (8) then yield \( \phi'(x_{2p} - x_{2p-1}) = 0 \) and hence

\[ x_{2p} = x_{2p-1} \]

as expected. In addition, substituting (5) and (7) in (3), retaining \( O(1) \) terms in the Taylor series and using (9), we recover Eq. (4) governing the slow dynamics:

\[ 2x_{2p} = \phi'(x_{2p+1} - x_{2p}) - \phi'(x_{2p-1} - x_{2p-2}) \]

Collecting now the \( O(\epsilon) \) terms in (8) and using (9) and \( \phi'(0) = 1 \), we obtain

\[ y_{2p} + y_{2p-1} = \phi(x_{2p+1} - x_{2p}) \]

and

\[ y_{2p-1} + y_{2p-2} = \phi(x_{2p} - x_{2p-1}) - x_{2p} \]

Note, however, that the last equation is equivalent to (11) due to (7), (9), (10), and thus is redundant.

Rewriting (11) in slow time and using (7), we obtain

\[ y_{2p} + \epsilon y_{2p} = \frac{1}{\epsilon}(\phi(x_{2p+1} - x_{2p}) - x_{2p}) \]

a system of decoupled linear oscillators with the forcing term due to the slow dynamics. Recall now that the latter is governed by (10), which implies that \( x_{2p} = x_{2p-1} = X_p(t) \), where \( X_p(t) \) satisfies the FPU equation for the double mass:

\[ 2x_{p} = \phi'(x_{p+1} - x_{p}) - \phi'(X_{p} - X_{p-1}) \]

In what follows we assume that (13) has a traveling wave (kink) solution

\[ x_p = X(\xi_0), \xi_0 = p - c_0 t \]

which tends to constant values at infinity and corresponds to a solitary pulse wave in terms of strain or particle velocity. This must hold if the potential \( \phi(\xi) \) satisfies the assumptions of [31]. Here \( c_0 \) is the velocity of the wave (which must be supersonic), and \( \xi_0 \) is the traveling wave coordinate. The function \( X(\xi_0) \) then satisfies the advance-delay differential equation

\[ 2c_0^2 X''(\xi_0) = \phi'(X(\xi_0 + 1) - X(\xi_0)) - \phi'(X(\xi_0) - X(\xi_0 - 1)) \]

We then have

\[ x_{2p} = x_{2p-1} = X(p - c_0 t) \]

and so the right hand side in (12) is

\[ \frac{1}{\epsilon} F(p - c_0 t) = \frac{1}{\epsilon} F(\xi_0) \]

\[ = \frac{1}{\epsilon} \left[ \phi'(X(\xi_0 + 1) - X(\xi_0)) - c_0^2 X''(\xi_0) \right] \]

Due to the invariance to translation in time and displacement, it is always possible to choose an odd kink solution \( X(\xi_0) \) of (15), so that \( X(\xi_0) = -X(-\xi_0) \), and therefore its second derivative is also odd. This implies that \( F(\xi_0) \) in (17) is even: \( F(\xi_0) = F(-\xi_0) \). Indeed, note that (15) implies that \( F(\xi_0) = c_0^2 X''(\xi_0) + \phi'(X(\xi_0) - X(\xi_0 - 1)) \), so that we have

\[ F(-\xi_0) = \phi(X(-\xi_0 - 1) - X(-\xi_0)) - c_0^2 X''(\xi_0) \]

where we used the fact that both \( X(\xi_0) \) and \( X'(-\xi_0) \) are odd. Note also that it suffices to consider (12) at \( p = 0 \), since the other \( y_{2p}(t) \) can be recovered from \( y_0(t) \) using time shifts. We thus have

\[ y_0 + \frac{2}{\epsilon} y_0 = \frac{1}{\epsilon} f_0(t), \]

where

\[ f_0(t) = F(-c_0 t) \]

has an even symmetry about \( t = 0, f_0(t) = f_0(-t) \), by the above.

Seeking solution of (18) that satisfies \( y_0 \to 0 \) as \( t \to -\infty \), we obtain

\[ y_0(t; \epsilon) \sim \frac{1}{\sqrt{2\epsilon}} \left[ \sin \left( t \sqrt{\frac{2}{\epsilon}} \right) \int_{-\infty}^{t} f_0(s) \cos \left( s \sqrt{\frac{2}{\epsilon}} \right) ds \right. \]

\[ - \cos \left( t \sqrt{\frac{2}{\epsilon}} \right) \int_{-\infty}^{t} f_0(s) \sin \left( s \sqrt{\frac{2}{\epsilon}} \right) ds \] .

Since \( f_0(t) \) is even and \( \sin(t, \sqrt{2/\epsilon}) \) is odd, the second integral vanishes in the limit \( t \to -\infty \), and we have the following asymptotic behavior of \( y_0 \):

\[ y_0(t; \epsilon) \sim \frac{1}{\sqrt{2\epsilon}} \sin \left( t \sqrt{\frac{2}{\epsilon}} \right) \int_{-\infty}^{t} f_0(s) \cos \left( s \sqrt{\frac{2}{\epsilon}} \right) ds \]

\[ t \to -\infty. \]

Thus \( y_0(t, \epsilon) \) exhibits oscillations at infinity, with frequency \( \omega = \sqrt{2/\epsilon}, unless \epsilon \) is such that

\[ g(\epsilon) = \int_{0}^{\infty} f_0(s) \cos \left( s \sqrt{\frac{2}{\epsilon}} \right) ds = 0. \]

This orthogonality condition determines the asymptotic approximation of the antiresonance values of \( \epsilon \) at which we expect solitary wave solutions without oscillations at infinity.

Following [28], where a similar condition arises for a related class of problems, we now assume that \( f(s) \) can be continued into the complex plane and that its singularities (poles or branch points) in the upper half plane closest to the real axes are \( s_{\pm} = \pm \alpha \pm i\beta \), where \( \alpha > 0, \beta > 0 \). Near the singularities it has the form

\[ f_0(s) \sim C_\epsilon e^{\epsilon / \rho / 2 (s - s_{\pm})^2}, s \to s_{\pm} \]

where \( C_\epsilon \equiv C_{\epsilon, \epsilon} \), and \( \rho \) is a real number not equal to a non-negative integer. Then [28]

\[ g(\epsilon) = \frac{2\pi|C_\epsilon| e^{-\rho \sqrt{2/\epsilon}}}{F(-\rho)} \left( \frac{\epsilon}{2} \right)^{(\rho + 1)/2} \cos \left( \sqrt{\frac{2}{\epsilon}} + \delta \right) \].
where $\delta = \arg(C_i)$. This implies that $g(\epsilon) = 0$ has an infinite number of roots $\epsilon_n$ at which $\alpha \sqrt{2/\epsilon_n} + \delta = \pi/2 + \pi n$, for integer $n$. These roots accumulate at zero when $n \to \infty$. This general result is illustrated in the next section by a detailed calculation for a Toda lattice with alternating bonds.

4. Example: lattice with alternating Toda potentials

As a specific example illustrating our analysis, we now consider the (rescaled) Toda potential
\begin{equation}
    \phi(r) = r + e^{-r} - 1,
\end{equation}
which satisfies all of our assumptions. In this case the exact odd solution of (15) is given by [29]}
\begin{equation}
    X(\xi_0) = \kappa + \ln \frac{1 + \exp[2\kappa (\xi_0 - \frac{1}{2})]}{1 + \exp[2\kappa (\xi_0 + \frac{1}{2})]},
\end{equation}
with
\begin{equation}
    c_0 = \frac{\sinh \kappa}{\sqrt{2\kappa}}.
\end{equation}
The parameter $\kappa > 0$ thus controls the speed of the wave; when $\kappa \to 0$, we approach the sonic limit. This yields
\begin{equation}
    F(\xi_0) = -\frac{2(1 + \cosh \kappa \cosh(2\kappa \xi_0))\sinh^2 \kappa}{(\cosh \kappa + \cosh(2\kappa \xi_0))^2},
\end{equation}
and hence
\begin{equation}
    f_0(t) = F(-c_0 t) = -\frac{2(1 + \cosh \kappa \cosh(\sqrt{2}(\sinh \kappa \xi))\sinh^2 \kappa}{(\cosh \kappa + \cosh(\sqrt{2}(\sinh \kappa \xi))^2}.
\end{equation}
In view of (22), the problem of finding asymptotic antiresonance $\epsilon$ values reduces to evaluating zeroes of
\begin{equation}
    g(\epsilon) = -\Re \int_{-\infty}^{\infty} \Lambda(z) \exp \left( -\frac{iz}{\sqrt{\epsilon} \sinh \kappa} \right) dz,
\end{equation}
where we introduced the new variable $z = t \sqrt{2} \sinh \kappa$ and the function $\Lambda(z)$ given by
\begin{equation}
    \Lambda(z) = \frac{(1 + \cosh \kappa \cosh z) \sinh \kappa}{\sqrt{2}(\cosh \kappa + \cosh z)^2}.
\end{equation}
The integral in (27) is readily evaluated using contour integration in the complex plane. Note that $\Lambda(z)$ has poles of order 2 at $z = z_\pm + 2\pi \i n i$, for any integer $n$, where
\begin{equation}
    z_\pm = \pm \kappa + \i \pi.
\end{equation}
We now extend $z$ in (27) to the complex plane and consider the integral along the rectangular contour $C_0$, oriented counterclockwise, which consists of the path along the real axis from $z = -R$ to $z = R$, a path parallel to the real axis from $z = R + 2\pi i$ to $z = -R + 2\pi i$ and two side paths parallel to the imaginary axis that close the contour. Note that for $R > \kappa$ the contour $C_0$ encloses two poles of $\Lambda(z)$, at $z = z_\pm$. One can show that in the limit $R \to \infty$ the contributions to the integral along the side paths vanish. Considering the contribution due to the remaining parts in this limit and using the residue theorem, we obtain
\begin{equation}
    \left( 1 - e^{-\frac{2\pi}{\sqrt{\epsilon} \sinh \kappa}} \right) \int_{-\infty}^{\infty} \Lambda(z) \exp \left( -\frac{iz}{\sqrt{\epsilon} \sinh \kappa} \right) dz
    = 2\pi i \left( \text{Res}(\Lambda(z)e^{\sqrt{\epsilon} \sinh \kappa})|_{z = z_-} + \text{Res}(\Lambda(z)e^{\sqrt{\epsilon} \sinh \kappa})|_{z = z_+} \right).
\end{equation}

Evaluating the residues in the right hand side and simplifying, we obtain the following simple expression for $g(\epsilon)$:
\begin{equation}
    g(\epsilon) = -\pi \sqrt{2} \cos \left( \frac{\kappa \cosh(\sqrt{\epsilon} \sinh \kappa)}{\sqrt{\epsilon}} \right) \cosh \left( \frac{\pi \cosh(\sqrt{\epsilon} \sinh \kappa)}{\sqrt{\epsilon}} \right).
\end{equation}
Note that this is consistent with (23): in this case we have $\alpha = \kappa \cosh(\sqrt{\epsilon} \sinh \kappa)/\sqrt{2}$, $\beta = \pi \cosh(\sqrt{\epsilon} \sinh \kappa)/\sqrt{2}$, $\rho = -2$ and $\delta = \pi$. Eq. (30) yields the following sequence of (asymptotic) antiresonance values of $\epsilon$ as functions of $\kappa$ (which, as we recall, controls the wave speed):
\begin{equation}
    \epsilon_n(\kappa) = \frac{k^2 \cosh^2(\sqrt{\epsilon} \sinh \kappa)}{\pi^2(n + \frac{1}{2})}, \quad n = 0, 1, 2, \ldots
\end{equation}
As illustrated in Fig. 3, for a given $n$, $\epsilon_n(\kappa)$ decreases as $\kappa$ grows.

We now compare these asymptotic results with numerical simulations. To this end, we solved the system (1) on the truncated lattice of 400 particles with zero-strain boundary conditions and initial conditions determined from the slow dynamics (16), (25) for given $\kappa$. As shown in Fig. 1, for generic values of $\epsilon$, this results in pulses radiating energy via oscillations. To find the antiresonance $\epsilon$ values at which these oscillations have practically zero amplitude (within numerical resolution), we minimized the energy stored at a fixed site $m = m_0$ at a time instant $t = t_0$ after the pulse has left the site. With this approach we were able to identify approximate $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ values for given $\kappa$. Since $\epsilon_n$ rapidly decrease as $n$ grows, and the energy of emitted oscillations is very small at any $\kappa$ close to zero (note the exponential decay in (30) as $\kappa \to 0$), it is difficult to find $\epsilon_n$ numerically using our approach when $n$ is sufficiently large. We were also unable to find an equivalent of $\epsilon_0$ seen in the asymptotic analysis; since the asymptotic $\epsilon_0$ is not much less than one (e.g. $\epsilon_0 \approx 0.12$ at $\kappa = 2$), the corresponding dynamics violates the separation of scales assumption, and this value should probably be discarded. The resulting velocity profiles for $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ at $\kappa = 2$ are shown in Fig. 4. Note that unlike solitary waves in lattices with uniform potentials, the velocity profiles in this case do not have even symmetry; however, the strain profiles (shown in Fig. 5 below) possess such symmetry.
Asymptotic and numerical antiresonance values for Toda lattice.

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5. Concluding remarks

In this work we considered a chain of alternating softer and stiffer bonds, with the ratio of the elastic moduli equal $\varepsilon$. We used multiple scale asymptotic analysis in the limit of small $\varepsilon$ (high mismatch between the bonds) to derive an orthogonality condition that yields an infinite sequence $\epsilon_n$ approximating the antiresonance modulus ratios at which the amplitude of the optical oscillations emitted by a moving pulse vanishes, and a solitary wave propagates through the chain without losing energy. Using Toda potential as an example, we found explicit formula for $\epsilon_n$ and approximate solutions that turned out to be in a very good agreement with numerical simulations. This result complements our recent work [22] on a different type of periodic heterogeneous lattice, with lighter and heavier masses alternate instead of bond potentials, where an orthogonality condition was also derived, and the antiresonance mass ratio values were computed for the Toda potential, as well as the earlier work [20,21] on the diatomic granular chains that has motivated this study. While the general approach used here is similar to the one pursued in this earlier work, the resulting waveforms are quite different in the two cases. Interestingly, the analysis turned out to be simpler in the present setting, since the fast dynamics in this case has constant frequency and involves sines and cosines instead of hypergeometric functions that arise in [22].

It should be noted that while our example in Section 4 takes advantage of the full integrability of the homogeneous Toda lattice to derive exact asymptotic values of $\epsilon_n$, the results of this work can also be used for other potentials, where the exact solitary wave solution in the $\varepsilon = 0$ case is not known but a good approximation is available instead, e.g. based on a quasicontinuum approximations near the sonic limit and global approximation of the discrete operator in Fourier space [32]. Substituting the corresponding approximation of $f_0(s)$ in (20) and (22) would then enable one to compute approximate $\epsilon_n$ and the resulting waveforms.
Although the work presented here focuses on the generic class of potentials with nonzero elastic moduli, the analysis can also be extended to the strongly nonlinear case of sonic vacuum, where the scaling is generally different and the oscillation frequency is time-dependent. An interesting example of such system was recently considered in [30] for a granular chain of cylinders stacked at alternating contact angles.

The ability to “tune” periodic heterogeneous systems to either enable (at antiresonance parameter values) or prevent (at resonance ones) energy transport through the lattice is clearly important in engineering applications. Understanding how heterogeneous systems in nature enable energy transport is also an important question in biophysics. In particular, solitonic excitations have been assumed to be responsible for energy transport practically without loss in muscle proteins, which can be modeled by periodic heterogeneous chains with alternating masses and bond potentials [14]. One of the interesting future directions is to extend the analysis presented here and in [22] to such lattices, using a two-parameter asymptotic expansion. Another challenging problem is establishing the existence of antiresonance parameter values in such systems in the regime beyond the asymptotic limit in which the fast and slow time scales are not well separated. Finally, it would be interesting to investigate whether these results can be extended to lattices with alternating bonds governed by nonconvex potentials.

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References