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What is This?
Stick-Slip Interface Motion as a Singular Limit of the Viscosity-Capillarity Model

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Abstract: This work is a follow-up on a study by Vainchtein and Rosakis of interface dynamics and hysteresis in materials undergoing solid-solid phase transitions. The author investigates the dynamics of a bar with a nonconvex double-well elastic energy density. The model includes both viscosity and strain-gradient capillarity terms. Viscous stress provides energy dissipation. The capillarity term models interfacial energy. The bar is subject to time-dependent displacement boundary conditions. Numerical simulations predict hysteretic behavior in the overall load-elongation diagram. The hysteresis is primarily due to metastability and persists even at very slow loading when viscous dissipation is small. At a given loading, a large capillarity coefficient results in a smooth interface motion and small hysteresis loop. As the coefficient becomes smaller, the loop grows and acquires serrations, while the interface motion alternates between slow and fast regimes. The results suggest that the stick-slip interface motion and serrated hysteresis loop predicted by Vainchtein and Rosakis in the absence of interfacial energy are a singular limit of the viscosity-capillarity model as the capillarity coefficient tends to zero. The irregular interface motion and serrated load-elongation curves qualitatively agree with some experimental observations in shape-memory alloys.

Key Words: phase boundary, stick-slip motion, hysteresis, viscoelasticity, higher-order gradients

1. INTRODUCTION

Materials undergoing stress-induced martensitic phase transitions commonly form various layered microstructures [1]. When subjected to cyclic loading, these materials exhibit hysteresis loops in the load-displacement curves which are often serrated [2-7]. The serrations are accompanied by phase nucleations and a nonsmooth, irregular motion of the phase boundaries [3, 4].

A common approach to explain these phenomena within the framework of elasticity theory involves minimization of a nonconvex elastic energy for the material [8-14]. The idea was introduced by Ericksen [15], who has studied equilibria of an elastic bar with a nonmonotone stress-strain curve. Basic features of the microstructure have been successfully captured by the absolute minimization of such potential energy [16]. However, this approach cannot account for hysteresis, a phenomenon associated with the material getting locked in local minimizers of the potential energy [17, 18].

The importance of local minimizers in the dynamics of phase transition has been demonstrated by the studies [10, 19, 20] of the dynamics of a viscoelastic Ericksen’s

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bar with a nonconvex double-well elastic energy density, placed on an elastic foundation and subjected to zero displacement boundary conditions. The statics analysis shows that the elastic foundation makes finer and finer phase mixtures energetically more favorable. However, numerical results in [10] show that the dynamic solutions typically tend to the states with a finite number of phase boundaries. These states locally minimize potential energy with respect to smooth perturbations that leave the locations of interfaces frozen. Following [21], we will refer to them as elastic minimizers. The dynamic stability of these minimizers was confirmed analytically in [19, 20]. Similar results were shown earlier in [22] for the viscoelastic bar with a nonmonotone stress-strain law subjected to a time-independent loading in a soft device. The finite scale of phase layering also agrees with experimental results [6, 7].

The results of these studies and a desire to study the overall hysteretic behavior have motivated the work [23], where the dynamics of a viscoelastic Ericksen’s bar with a double-well energy density subjected to time-dependent displacement-controlled loading was investigated. The elastic foundation term was not included; see, however, [24]. Each energy-density well represented a material phase. Inertia was taken into account. Viscous stresses, proportional to the strain rate, provided energy dissipation, while time-dependent displacement-controlled loading supplied the energy to the system. A stress-free single-phase equilibrium state was chosen as an initial condition. The bar was then subjected to cyclic loading at the ends. This provided a physically appropriate way to model tension tests in bars and facilitated the study of overall hysteresis loops in load-elongation curves.

As shown in [15], in this case potential energy possesses infinitely many elastic minimizers. Each minimizer has piecewise constant strain with a number of finite jump discontinuities. The discontinuities can be thought of as phase boundaries. These sharp interfaces cannot move [22]. It is shown in [23] that with sufficiently slow loading, the dynamic solution that started in a stress-free single-phase state follows closely a branch of metastable single-phase equilibria that emerge from the initial state, until the time-dependent loading causes the strain to enter the spinodal region, or the region where the elastic energy density is concave. The spinodal instability then results in formation of several sharp phase boundaries. After that, the solution undergoes two different regimes. In the stick regime, it stays on one of the quasistatic branches of elastic minimizers with fixed positions of the interfaces. While the phase boundaries do not move, the strain in the bar grows to adjust to the loading. The solution stays on the branch until the strain enters the spinodal region in part of the bar. In what is called a slip regime, the spinodal instability smoothes the strain profile and moves the interfaces to their new locations. This results in serrations, or “teeth,” on the overall load-elongation curve. Both serrations and the stick-slip interface motion have been experimentally observed in shape-memory alloys [2-5]. The overall shape of the loop is in qualitative agreement with these experimental results. The hysteresis is observed even at a very slow loading when viscosity effects are minor.

The present paper is a follow-up work on the study described in [23]. One of the questions left unanswered in [23] was the relationship between the viscoelastic Ericksen’s bar models with and without interfacial energy, modeled by a strain-gradient capillarity term $\alpha u^2_{xx}$. In the latter so-called viscosity-capillarity model, also considered in [23], the number of local minima becomes finite, and sharp interfaces are replaced by smooth transition layers (see, e.g., [25]). Such interfaces can move [26]. In this case, only single-phase and two-phase solution branches contain local and global minimizers of potential energy. It is shown in [23]
that when the interfacial energy is present and the loading is sufficiently slow, the dynamic solution switches to the lower-energy two-phase branch after the strain enters the spinodal region. It then follows the branch, while the interface is moving. No serrations are observed, and the hysteresis loop is in much less qualitative agreement with the experimental results than the model without the interfacial energy.

In this paper, we revisit the comparison and show that for a given loading, a sufficiently small interfacial energy coefficient $\alpha$ also results in the end-load serrations. This happens because the speed of the transition layers is proportional to $\sqrt{\alpha}$ [26], and one can always find small enough $\alpha$ at which the velocity of the interface is much smaller than the external loading rate. As a result, the strain in the bar grows to adjust to the loading, while the phase boundary moves very slowly. Meanwhile, the system goes away from the minimum-energy two-phase branch, with both potential energy and end load increasing. Eventually the strain enters the spinodal region in a portion of the bar, and the instability causes the system to lower the energy and adjust the location of the phase boundary. These results suggest that the stick-slip interface motion is a singular limit of the dynamic model with interfacial energy as $\alpha$ goes to zero.

The structure of the paper is as follows. We formulate the problem and describe the structure of equilibria in Section 2. In Section 3, we summarize the results of [23] and show how the dynamics of the model with interfacial energy approaches the stick-slip interface motion as the strain-gradient term becomes small.

2. PROBLEM FORMULATION AND EQUILIBRIA

Consider a bar of unit undeformed length, with constant reference density $\rho > 0$. Let $u(x, t)$ denote the longitudinal displacement field, where $x \in [0, 1]$ is the reference coordinate of a point along the bar and $t$ is time. The elastic strain energy density of the bar is given by $W(u_x)$, where $u_x$ is the deformation strain. The elastic energy of the bar equals $\int_0^1 W(u_x)dx$, where the energy density

$$W(u_x) = \frac{1}{4}(u_x^2 - 1)^2$$

is a double-well potential with minima at $u_x = \pm 1$. The two wells represent two different material phases. The energy density is convex in the intervals $I_+ = (-\infty, -1/\sqrt{3})$, $I_- = (1/\sqrt{3}, \infty)$ and nonconvex when strain is in the interval $I_0 = [-1/\sqrt{3}, 1/\sqrt{3}]$. See Figure 1.

We will refer to the intervals $I_+$ and $I_-$ as $+$ and $-$ phase intervals, respectively. The interval of nonconvexity $I_0$ is called spinodal region.

The static problem of minimization of the potential energy functional

$$E_{pot} = \int_0^1 W(u_x)dx$$

subject to the boundary conditions $u(0) = 0$ and $u(1) = d$ was considered in [15]. For $d$ in $[-1, 1]$, apart from the homogeneous equilibrium solution $u(x) = dx$, which is unstable
Fig. 1a. Energy density $W(u_x)$.

Fig. 1b. Stress $\sigma(u_x) \equiv W'(u_x)$. 
when $d$ is in $I_0$, there are infinitely many inhomogeneous equilibrium states. Following [21], we will call these states elastic equilibria. In these solutions, $u$ is continuous but the strain is piecewise constant, alternating between the values $e_1$ and $e_2$, satisfying

$$W'(e_1) = W'(e_2),$$

so that the stress is constant in the bar:

$$\sigma(u_x) \equiv W'(u_x) = S,$$

and

$$e_1 s_1 + e_2 (1 - s_1) = d,$$

where $s_1$ is the total length of the portion of the bar that has strain $e_1$. The finite jump discontinuities in strain represent phase boundaries. Their number and location are arbitrary.

The elastic equilibria locally extremize the potential energy (2) with respect to $W^{1,\infty}$ variations, with the norm

$$\|u\|_{1,\infty} = \|u\|_\infty + \|u_x\|_\infty, \quad \|u\|_\infty \equiv \text{ess sup}_{x \in [0,1]} |u(x)|$$

(both the variation and its derivative are small), that leave the locations of phase boundaries frozen. An elastic equilibrium that satisfies $W''(u_x) > 0$ at every point in the bar (neither $e_1$ nor $e_2$ is in a spinodal region) is a local minimizer of (2) in this sense and is called weakly stable; all other elastic equilibria are unstable, by Legendre’s necessary condition [27]. The strong energy minimizers are equilibria that locally minimize (2) with respect to $L^\infty$ variations (with the norm $\|u\|_\infty$ defined by (6)$_2$; only variations are small but their derivatives need not be) that allow perturbations of the interface locations. The strong minimizers, in addition to (3) and (5), satisfy the Weierstrass–Erdman corner conditions

$$[W(u_x) - u_x \sigma(u_x)]_{c_i} = 0$$

at each interface. Here, $[A]_{x_0}$ denotes the jump in function $A(x)$ across $x = x_0$ and $c_i$ are the locations of interfaces in the reference configuration.

For a fixed value of $s_1$ (the volume fraction of, say, lower-strain phase), one can construct a branch of elastic equilibria and express the stress in the bar $S$ as a (in general multivalued) function of $d$. Figure 2 shows several of the constant-$s_1$ curves on a static load-displacement diagram. Weakly stable and unstable portions of the branches are shown by solid and dashed curves, respectively. Each curve bifurcates from the homogeneous solution branch at the points where the latter loses stability ($d = \pm 1/\sqrt{3}$). Along the weakly stable portions, $e_1$ and $e_2$ are in + and − phases, respectively. We emphasize that one can construct an uncountable infinity of such branches by varying $s_1$ continuously from 0 to 1. Moreover, each point on a given $s_1$-curve corresponds to a family of solutions $(e_1, e_2, s_1)$ obtained by different arrangements of phases with strains $e_{1,2}$ at fixed volume fraction $s_1$. Thus, any point $(d, S)$ inside the shaded loop shown in Figure 3 corresponds to two families of elastic equilibria,
Fig. 2. The load-displacement diagram showing several branches of elastic equilibria with different fixed volume fraction $s_1$ of the lower-strain phase along each branch. Weakly stable and unstable portions are shown by the solid and dashed lines, respectively. The value of $s_1$ decreases from 1 to 0 from left to right.

Fig. 3. Every point inside the shaded loop on the static load-displacement diagram corresponds to two families of elastic equilibria, one of which consists of (weakly) stable solutions. The thick zero-load Maxwell line corresponds to absolute energy minimizers.
one of which consists of weakly stable solutions; the others are unstable. Each point on
the boundary of the loop corresponds to either a unique (weakly) stable single-phase elastic
equilibrium (right and left boundaries, \( d \) in \( l_{+} \)) or a family of marginally stable multiphase
solutions (upper and lower boundaries).

In strong minimizers, strain alternates between \(-1\) and \(1\) (the minima of the energy
density (1)). These minimizers globally minimize the potential energy in this case, with
zero stress. See the horizontal zero-load line in Figure 3, also called the Maxwell line. For a
general energy density, this line cuts equal areas above and below the graph of \( \sigma(u_x) \).

Another model considered in this paper also accounts for interfacial energy, modeled by
the strain-gradient term \( \int_0^1 \alpha u_x^2 \, dx \), with constant \( \alpha > 0 \). In this case, the potential energy
equals

\[
E_{pot} = \int_0^1 [W(u_x) + \alpha u_x^2] \, dx. \tag{8}
\]

The introduction of strain-gradient term has been widely used to analyze spinodal region
decomposition, phase transitions, and other phenomena, for example, [28-31]. The strain-
gradient term penalizes the formation of phase boundaries: in extremals of (8), strain-
discontinuities are replaced by smooth transition layers of width proportional to \( \sqrt{\alpha} \).

Minimizing (8) subject to displacement boundary conditions \( u(0) = 0 \) and \( u(1) = d \) results in the Euler-Lagrange equation

\[
\sigma'(u_x) u_{xx} - 2\alpha u_{xxxx} = 0 \tag{9}
\]

and natural boundary conditions \( u_{xx}(0) = u_{xx}(1) = 0 \). The strain-gradient term regularizes
the equilibrium equation, and weak local minimizers (in \( H^2 \) space) are also strong (in \( C^2 \)-
topology) in this case. A typical load-displacement diagram of equilibria is shown in Figure 4.

For each \( d \), there is a homogeneous solution branch with \( u(x) = dx \), with no transition
layers (\( n = 0 \)). From this main branch (shown by a thick curve), a finite number
of equilibrium branches (thin curves) bifurcate. Each branch contains nonhomogeneous
solutions with \( n \) transition layers that represent phase boundaries. The solutions satisfy
\( u_n(x) = dx + \varepsilon \sin(n\pi x) + o(\varepsilon) \) in small \( \varepsilon \)-neighborhoods of the bifurcation points \( d_n \); for
a more complete description, see [25]. Among these, only \( n = 1 \) branch contains (absolute)
energy minimizers; the other \( n \neq 0 \) branches contain only unstable solutions [28]. As \( \alpha \) tends
to zero, the bifurcation points tend to cluster at the points \( d = \pm 1/\sqrt{3} \). The branches with
nonzero \( n \) also come closer to each other, with the stable part of \( n = 1 \) branch approaching
the Maxwell line.

We consider the dynamics of the bar, both with and without interfacial energy. The
viscous stress is linearly proportional to the strain rate \( u_{str} \), with constant viscosity coefficient
\( \gamma > 0 \). This term introduces energy dissipation. The total stress in the bar is given by

\[
\Sigma = \sigma(u_x) - 2\alpha u_{xxx} + \gamma u_{str}, \tag{10}
\]

where \( \sigma(u_x) \equiv W'(u_x) \) is the elastic contribution to the stress; the term \(-2\alpha u_{xxx}\) is due to the
strain-gradient term. The bar is subject to time-dependent displacement-controlled boundary
conditions. The dynamics of the bar is then described by the following initial-boundary-value problem:

\[
\begin{align*}
\rho \, u_{tt} &= \left[ \sigma(u_x) - 2au_{xxx} + \gamma u_{xt} \right]_x \\
\quad u(0, t) &= d_0(t) \\
\quad u(1, t) &= d_1(t) \\
\quad au_{xx}(0, t) &= au_{xx}(1, t) = 0 \\
\quad u(x, 0) &= u_0(x) \\
\quad u_t(x, 0) &= 0.
\end{align*}
\]

Initially, the bar is in a stable stress-free equilibrium state \( u_0(x) \) with zero initial velocity. The time-dependent boundary conditions are chosen to be either symmetric,

\[
d_0(t) = -d(t)/2, \quad d_1(t) = d(t)/2,
\]

or nonsymmetric,

\[
d_0(t) = 0, \quad d_1(t) = d(t).
\]
Here, \( d(t) \) is the loading function. In our numerical examples, we will employ the following cyclic loading function:

\[
d(t) = \begin{cases} 
(\Delta_T - \Delta) \frac{t^2}{t_i t_r} + \Delta & \text{for } 0 \leq t \leq t_i \\
(\Delta_T - \Delta) \left[ -\frac{(t - t_i)^2}{t_r (t_r - t_i)} + 2 \frac{t}{t_r} - \frac{1}{t_r} \right] + \Delta & \text{for } t_i \leq t \leq 2t_r - t_i \\
(\Delta_T - \Delta) \frac{(t - 2t_r)^2}{t_i t_r} + \Delta & \text{for } 2t_r - t_i \leq t \leq 2t_r.
\end{cases}
\]  

(14)

Here, \( \Delta, \Delta_T, t_i, t_r \) are constants. Graphs of the loading function \( d(t) \) and the loading rate \( d'(t) \) are shown in Figure 5.

The initial end displacement is \( d(0) = \Delta \). Loading occurs with \( d \) increasing up to a maximum \( d(t_r) = \Delta_T \) at \( t = t_r \), followed by unloading to the initial value \( d(2t_r) = \Delta \). The parameter \( t_r \) is inversely proportional to the loading rate amplitude. Parameter \( t_i \) controls the initial acceleration \( d''(0) \).

One can show that the rate of change of total (potential plus kinetic) energy is equal to

\[
E'(t) = S(t)d'(t) - \gamma \int_0^1 u_{st}^2 \, dx. 
\]  

(15)

In case of nonsymmetric boundary conditions (13), \( S(t) \) equals the end load \( \Sigma(1, t) \) (recall (10)). When symmetric boundary conditions (12) are chosen, it equals

\[
S(t) = \frac{1}{2} \left[ \Sigma(0, t) + \Sigma(1, t) \right].
\]  

(16)

The term \( S(t)d'(t) \) in (15) is the loading power; it supplies energy into the system. The second term, \( \gamma \int_0^1 u_{st}^2 \, dx \), in (15) is the energy dissipation rate caused by the presence of viscous stresses.

Since the strain-gradient term is also used to describe capillary phenomena, dynamic models involving both viscosity and strain-gradient terms are sometimes called viscosity-capillarity models (see, e.g., [26]).

In the special case of \( \alpha = 0 \) (but viscosity and inertia terms still present) considered in [23], the stress is \( \Sigma = \sigma(u_x) + \gamma u_{xt} \) and the initial-boundary-value problem reduces to

\[
\begin{align*}
\rho \ u_t &= [\sigma(u_x) + \gamma u_{xt}]_x \\
u(0, t) &= -d(t)/2 \\
u(1, t) &= d(t)/2 \\
u(x, 0) &= u_0(x) \\
u_t(x, 0) &= 0.
\end{align*}
\]  

(17)
Fig. 5a. Cyclic loading function $d(t)$ used in numerical simulations.

Fig. 5b. Loading rate $d'(t)$. 

3. NUMERICAL RESULTS AND DISCUSSION

The numerical solution of (17) was studied in [23] using an implicit finite-difference scheme adapted from [32]. In this section, we summarize the result. We also show that it can be seen as a singular limit of the viscosity-capillarity model.

The cyclic loading (14) in this case results in the hysteresis loop shown in Figure 6 where the end load \( S \) defined by (16) is plotted against the end displacement \( d \).

The most interesting feature of the loop is that it possesses a number of serrations. Such "teeth" are often experimentally observed [2, 5]. The simulation starts in a stress-free equilibrium configuration in the + phase with \( u_+ = -1 \). Initially, the dynamic solution is close to the equilibria with the whole bar in the + phase. Upon reaching the maximum of the first "tooth," the strain enters spinodal region where the static solution is unstable. As a result, the strain gradient grows and, eventually, two sharp phase boundaries form in the middle of the bar. The interface formation is accompanied by a drop in end load (dash-dotted curve). Now the bar is occupied by the – phase in the middle and the + phases at the ends, with strain discontinuities across each phase boundary. As the loading is continued, the newly formed boundaries do not move but the strain in the regions separated by the interfaces grows. This was referred to in [23] as a stick regime. It corresponds to a portion of the second tooth where stress grows (solid curve). Eventually, the strains at the ends of the bar enter spinodal region, and instability again increases the strain gradient until the strain profile smoothens. The old discontinuities are now destroyed and the new ones are formed closer to the ends of the bar. This is called a slip regime (dashed curve), and during this regime the stress drops. This process of alternating stick and slip regimes happens several times until the whole bar is in the – phase.

It was shown in [23] that during a stick regime dynamic solution is close to a branch of elastic equilibria with fixed positions of the interfaces. Several of the infinite continuum of these branches are shown in Figure 2. Recall that along each curve the volume fraction \( s \) of + phase is fixed. Figure 7 from [23] compares the end load curve from the dynamical simulations during loading with the quasistatic constant-\( s \) curves.

During each stick regime, the end load closely follows one of the \( s \)-curves while the interfaces are stationary. When strain in a portion of the bar is sufficiently well within the spinodal region (past the \( s \)-curve maximum) for the instability to initiate a slip event, the dynamic solution switches to a different \( s \)-curve, and the end load drops.

In [23], the model of viscoelastic Ericksen’s bar was compared to the viscosity-capillarity model given by (11), \( \alpha > 0 \), and boundary conditions (13). It was shown that at a slow loading the models with zero and nonzero \( \alpha \) predict different dynamics of the interfaces. At \( \alpha > 0 \), a phase boundary moves smoothly. When the interfacial energy term is omitted, the interfaces move in the stick-slip fashion described above, and hysteresis loop is much more pronounced and exhibits serrations.

We will now show the numerical results that suggest that the stick-slip interface motion is actually a singular limit of the problem (11) at a given loading function \( d(t) \). The finite-difference scheme from [33] was adapted for these simulations.

Consider the branches of equilibria of the problem (11) with the end displacement \( d \) viewed as a parameter. Figure 8 displays equilibrium branches relevant here. Among these is the \( n = 0 \) homogeneous-solution branch, which is unstable between points \( A \) and \( D \) and stable (contains local potential energy minimizers) elsewhere. The other relevant branch is
Fig. 6. End load $S$ versus $d$ during a loading-unloading cycle from a numerical solution of problem (17), without interfacial energy ($\alpha = 0$): $\gamma = 0.1, \rho = 0.05, t_T = 100$. Solid, dash-dotted, and dashed curves indicate stick, nucleation, and slip regimes, respectively. The asymmetry of the loop with respect to the reversion of the loading-unloading path is due to the asymmetric loading rate $d'(t)$ (see Fig. 5b). From [23].

Fig. 7. Comparison of dynamic (thick solid line, $\gamma = 0.1, \rho = 0.05, t_T = 100$) and quasistatic solutions (thin solid and dotted lines): end load $S$ versus $d$. The volume fractions of the $+$ phase from left to right are $s = 1, 0.74, 0.58, 0.45, 0.38, 0.25, 0.17, 0.1$. The values of $s$ used to calculate quasistatic branches were extracted from measurement of interface positions during stick regime in the dynamic solution. The dynamic end load is slightly larger than the quasistatic one due to the viscous stress term $\gamma \dot{\epsilon}$. From [23].
Fig. 8a. Diagram of potential energy $E$ versus end displacement $d$ for the $n = 0$ (dotted line) and $n = 1$ (solid line) equilibrium branches (bar with interfacial energy): $\alpha = 0.0005$. Part $AD$ of the $n = 0$ branch is unstable, the rest is stable. Parts $AB$ and $CD$ of the $n = 1$ branch are unstable, $BC$ is stable.

Fig. 8b. Stress $S$ versus end displacement $d$. From [23].
Fig. 9a. Potential energy $E$ versus $d$ from numerical solutions to dynamics problem (11) at different $t_T$ (larger $t_T$ means slower loading) shown by thick curves over the statics $n = 0$ (dotted line) and $n = 1$ (solid line) equilibrium branches: $\alpha = 0.0005$, $\gamma = 0.1$, $\rho = 0.05$, $t_i = 0.01$. Only the loading part is shown ($t < t_T$ in (14)).

Fig. 9b. End load $S(d)$. 
$n = 1$ branch $ABCD$, which contains stable solutions on the part $BC$ [28]. These solutions have one phase boundary.

Now consider numerically computed dynamic solutions of system (11), with the bar initially at the stress-free global minimum of the $+$ phase and boundary conditions given by (11)$_{2,3}$, (13), and (14). At sufficiently slow loading (large $t_R$ in (14)), as described in [23], the dynamic solution initially follows the static $n = 0$ branch quite closely. It passes the bifurcation point $A$ in Figure 8 and enters the statically unstable part of the branch, with strains in the spinodal region. At some point, both end load and potential energy drop close to the stable $n = 1$ branch, as shown in Figure 9 (thick curves depict dynamic solutions at different values of $t_R$). The drop is accompanied by a sudden nucleation of a finite interval of the $-$ phase, which occurs at one end of the bar, resulting in a solution with one phase boundary. The right and left ends of the bar are in the $+$ and $-$ phase, respectively, and the phases are separated by a "thick" phase boundary (transition layer) within which strain is in the spinodal region. The slower the loading (the higher $t_R$), the closer is the value of $d$ at which the nucleation occurs to $d_1$ (the value of $d$ at the bifurcation point $A$). This can also be seen from the linearized dynamics analysis [23].

If the loading is slow ($t_R = 1000$ and 100 shown by the thick solid and thick dashed curves in Figure 9, respectively), the solution then follows the $n = 1$ branch, as the interface moves smoothly to the right. When this branch no longer exists for the current value of $d$ (past the turning point $C$), solution drops onto the $n = 0$ branch again. At this point, the entire bar has strain in the $-$ phase.

Let us now examine what happens if we increase the loading rate (solutions at $t_R = 25$ and 5, shown by the dash-dotted and dash-double-dotted thick curves in Figure 9). Until the drop in potential energy and the end load occurs, the behavior of solution is similar, except that the end load is higher due to the viscous effects (note that the potential energy follows the static $n = 0$ branch) and the nucleation occurs at larger $d$. The end load has an initial peak due to the initial positive acceleration at $t \leq t_1$, which is much higher at these values of $t_R$ ($t_1$ is kept the same in all loadings). After coming close to the static $n = 1$ branch, however, the potential energy grows away from it, reaches a maximum, and gradually decreases again, dropping to $n = 0$ branch at large $d$.

The reason for an increase in potential energy (and the end load) after the nucleation lies in the fact that at these values of $t_R$ the speed of interface propagation is slower than the loading rate $d'(t)$. To get an idea of how fast the transition layers can move, consider the traveling wave solutions $u(x - Vt)$ of (11) on an infinite bar. Suppose that such a solution has strains $w_-$ and $w_+$ at the ends of the bar separated by a transition layer. One can show [26] that for a quartic energy density (1) the velocity of the traveling wave is given by

$$V = \frac{3\sqrt{2\sqrt{\alpha \mu}}}{\gamma} \left( \frac{w_+ + w_-}{2} \right), \quad (18)$$

where $\mu = 2$ is the elastic modulus at the minima of the energy wells. So the smaller $\alpha$ is, the slower the transition layers can move. When the loading rate is sufficiently high at a given $\alpha$, the transition layers cannot "catch up" with the loading and adjust the location of the phase boundary to the increasing average strain. Therefore, while the boundaries continue to move slowly, the strains on either side of the boundary also grow to satisfy the boundary conditions. This causes growth of both potential energy and the end load. Note
that as the average strain grows, so does the velocity of the transition layer in (18), since both \( w_+ \) and \( w_- \) increase. Thus, the transition layer starts to move faster and is better able to keep up with the loading rate, which is also getting smaller according to (14). In the meantime, kinetic energy and the dissipation rate \( \gamma \int_0^1 u_x^2 \, dx \) have grown due to the increase in the velocity of the layer. The total energy rate is given by (15), with the end load now given by \( S(t) = \sigma(u_x'(1,t)) + \gamma u_x'(1,t) - 2au_{xxx}(1,t) \). When the dissipation rate exceeds the loading power, the total energy rate becomes negative, so both potential and kinetic energies decrease and the system gets closer to the \( n = 1 \) static branch. At larger \( d \), the loading rate becomes small and the system approaches the \( n = 0 \) branch.

If, instead of (13), symmetric boundary conditions (12) are prescribed, the results of simulations are similar, except that the single-interface solutions are replaced by solutions with two phase boundaries (symmetric with respect to the center of the bar), with a – interval in the middle and + intervals at the ends. This happens because symmetric boundary conditions (12) introduce the symmetry \( u(x) = -u(1-x) \) into the problem. Since the initial condition \( u_0(x) = -x \) possesses this symmetry, so does the unique (see Theorem 1 in [23]) solution of the initial-boundary-value problem.

Let us now fix \( t_f = 25 \) and consider different small values of \( \alpha \). To facilitate the comparison with results of [23], consider symmetric boundary conditions (12). See Figure 10. At \( \alpha = 10^{-4} \) (dash-dotted curve), the loading rate is high enough to cause a barely visible second maximum in potential energy with solution behavior as just described. As we decrease \( \alpha \) to \( 10^{-5} \) (solid curve), this behavior becomes more pronounced since the boundaries now move even slower compared to the loading rate. During the nucleation, the potential energy drops to a lower value because at smaller \( \alpha \) the static \( n = 2 \) branch has lower energy. Observe that the second tooth in the end load has now grown bigger. When \( \alpha \) is decreased to \( 10^{-6} \) (dotted curve), the boundaries are so slow compared to the loading that the average strain continues to increase until strain gets into spinodal region in a portion of the bar. The resulting instability causes the system to drop closer to \( n = 2 \) branch and adjust the position of each interface and the values of strain on either side of it. Both potential energy and the end load drop. The process then repeats itself. Thus, three teeth form in this case. Finally, when \( \alpha = 0 \) (dashed curve), the boundaries, which are now represented by strain discontinuities, remain stationary (observe that \( V \) goes to zero in (18) as \( \alpha \) tends to zero) until smoothened by spinodal instability, and the stick-slip behavior, described in [23] and summarized above, results, with five teeth in this case.

These results suggest that at a given dynamic loading the stick-slip boundary motion is really a singular limit of a viscosity-capillarity model (11). The quasistatic \( (t_f \rightarrow \infty) \) limits, however, are different at \( \alpha > 0 \) and \( \alpha = 0 \), with the hysteresis loop shrinking in one case and increasing in another. See Figure 11.

When \( \alpha = 0 \), slower loading results in more but shallower teeth as explained in [23], with the boundaries moving a shorter distance during each slip event. As a result, the hysteresis loop becomes larger, and we conjecture that its limiting shape is the unserrated loop shown in Figure 11. This is the loop bounding the shaded area in Figure 3. In contrast, quasistatic loading at positive \( \alpha \) (no matter how small) results in the hysteresis only within ranges of metastability of the \( n = 0 \) branch. The reason for this difference is the difficulty of defining the quasistatic regime in the \( \alpha = 0 \) case. When \( \alpha \) is nonzero, in addition to a time scale, \( \gamma / \mu \), there is a length scale, \( \sqrt[3]{\alpha / \mu} \). This gives a velocity scale, \( \sqrt[3]{\alpha \mu / \gamma} \). When an external
Fig. 10a. Potential energy $E$ versus $d$ from numerical solutions to dynamics problem (11) at different $\alpha$: $\gamma = 0.1$, $\rho = 0.05$, $t_T = 25$, $t_i = 0.01$, loading only.

Fig. 10b. End load $S(d)$. 
Fig. 11. The hysteresis loops at zero and nonzero $\alpha$ at slower loading. Here $\alpha = 0$ and 0.0005, $\gamma = 0.1$, $\rho = 0.05$, $t_i = 0.01$, larger $t_T$ means slower loading.

loading rate is much less than this scale, the loading should be considered quasistatic. When $\alpha$ becomes zero, the length scale is lost and there is no velocity scale (the acoustic speed is not constant and does not provide a scale). Hence, there is nothing to compare the external loading rate to in this case, and the quasistatic regime is not well-defined. We emphasize, however, that at any given slow loading rate, there is serrated hysteresis in both the no-capillarity case and the case of a small enough nonzero $\alpha$.

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