Using row reduction to calculate the inverse and the determinant of a square matrix

Notes for MATH 0290 Honors by Prof. Anna Vainchtein

1 Inverse of a square matrix

An $n \times n$ square matrix $\mathbf{A}$ is called *invertible* if there exists a matrix $\mathbf{X}$ such that

$$\mathbf{AX} = \mathbf{XA} = \mathbf{I},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. If such matrix $\mathbf{X}$ exists, one can show that it is *unique*. We call it the *inverse of* $\mathbf{A}$ and denote it by $\mathbf{A^{-1}} = \mathbf{X}$, so that

$$\mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$$

holds if $\mathbf{A^{-1}}$ exists, i.e. if $\mathbf{A}$ is invertible. *Not all matrices are invertible.* If $\mathbf{A^{-1}}$ does not exist, the matrix $\mathbf{A}$ is called *singular* or *noninvertible*.

Note that if $\mathbf{A}$ is invertible, then the linear algebraic system

$$\mathbf{Ax} = \mathbf{b}$$

has a *unique* solution $\mathbf{x} = \mathbf{A^{-1}b}$. Indeed, multiplying both sides of $\mathbf{Ax} = \mathbf{b}$ on the left by $\mathbf{A^{-1}}$, we obtain

$$\mathbf{A^{-1}Ax} = \mathbf{A^{-1}b}.$$

But $\mathbf{A^{-1}A} = \mathbf{I}$ and $\mathbf{Ix} = \mathbf{x}$, so

$$\mathbf{x} = \mathbf{A^{-1}b}$$

The converse is also true, so for a square matrix $\mathbf{A}$,

$\mathbf{Ax} = \mathbf{b}$ *has a unique solution if and only if* $\mathbf{A}$ *is invertible.*

2 Calculating the inverse

To compute $\mathbf{A^{-1}}$ if it exists, we need to find a matrix $\mathbf{X}$ such that

$$\mathbf{AX} = \mathbf{I} \tag{1}$$

Linear algebra tells us that if such $\mathbf{X}$ exists, then $\mathbf{XA} = \mathbf{I}$ holds as well, and so $\mathbf{X} = \mathbf{A^{-1}}$.

Now observe that solving (1) is equivalent to solving the following linear systems:

$$\begin{aligned}\mathbf{Ax_1} &= \mathbf{e_1}\\ \mathbf{Ax_2} &= \mathbf{e_2}\\ &\cdots\\ \mathbf{Ax_n} &= \mathbf{e_n},\end{aligned}$$

where $\mathbf{x_j}$, $j = 1, \ldots, n$, is the (unknown) jth column of $\mathbf{X}$ and $\mathbf{e_j}$ is the jth column of the identity matrix $\mathbf{I}$. If there is a unique solution for each $\mathbf{x_j}$, we can obtain it by using elementary row operations to reduce the augmented matrix $[\,\mathbf{A} \mid \mathbf{e_j}\,]$ as follows:

$$[\,\mathbf{A} \mid \mathbf{e_j}\,] \longrightarrow [\,\mathbf{I} \mid \mathbf{x_j}\,].$$

Instead of doing this for each j, we can row reduce all these systems simultaneously, by attaching *all* columns of $\mathbf{I}$ (i.e. the whole matrix $\mathbf{I}$) on the right of $\mathbf{A}$ in the augmented matrix and obtaining *all* columns of $\mathbf{X}$ (i.e. the whole inverse matrix) on the right of the identity matrix in the row-equivalent matrix:

$$[\,\mathbf{A} \mid \mathbf{I}\,] \longrightarrow [\,\mathbf{I} \mid \mathbf{X}\,].$$

If this procedure works out, i.e. if we are able to convert $\mathbf{A}$ to identity using row operations, then $\mathbf{A}$ is invertible and $\mathbf{A^{-1}} = \mathbf{X}$. If we *cannot* obtain the identity matrix on the left, i.e. we get a row of zeroes, then $\mathbf{A^{-1}}$ does *not* exist and $\mathbf{A}$ is *singular*.

Example 1. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

or show that it does not exist.

Solution:

form the augmented matrix $[\,\mathbf{A} \mid \mathbf{I}\,]$: $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$

$R_3 - 3R_1 \longrightarrow R_3$: $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$

$R_2 - 2R_1 \longrightarrow R_2$: $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$

interchange R_2 and R_3 : $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$

$$R_2 \cdot (-1), \quad R_3 \cdot (-1): \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$R_2 - 3R_3 \longrightarrow R_2: \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$R_1 - 2R_2 \longrightarrow R_1: \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 7 & -6 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$R_1 - 3R_3 \longrightarrow R_1: \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

So

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Example 2. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{bmatrix}$$

or show that it does not exist.

Solution:

$$\text{form the augmented matrix } [\ \mathbf{A} \mid \mathbf{I}\ ]: \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 1 & -2 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 - R_1 \longrightarrow R_3: \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 + 2R_1 \longrightarrow R_2: \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{array}\right]$$

$$R_2/5, \quad R_3/(-4): \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2/5 & 1/5 & 0 \\ 0 & 1 & 2 & 1/4 & 0 & -1/4 \end{array}\right]$$

$$R_3 - R_2 \longrightarrow R_3: \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & -3/20 & -1/5 & -1/4 \end{array}\right]$$

The row of zeroes on the left means we cannot get the identity matrix there, and thus $\mathbf{A}$ is singular (no inverse exists).

Applying this procedure to an arbitrary 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we obtain (check!)

$$\mathbf{A^{-1}} = \frac{1}{\det\mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$\det\mathbf{A} = ad - bc,$$

provided that $\det\mathbf{A} \neq 0$. Otherwise, the inverse does not exist. In general, it is true that

$$\mathbf{A} \textit{ is invertible if and only if } \det\mathbf{A} \neq 0.$$

You can check that in the Example 2 above $\det\mathbf{A} = 0$.

3 Calculating determinants using row reduction

We can also use row reduction to compute large determinants. The idea is to use elementary row operations to reduce the matrix to an upper (or lower) triangular matrix, using the fact that

Determinant of an upper (lower) triangular or diagonal matrix equals the product of its diagonal entries.

As we row reduce, we need to keep in mind the following **properties of the determinants**:

1. $\det\mathbf{A} = \det\mathbf{A^T}$, so we can apply either row or column operations to get the determinant.
2. If two rows or two columns of $\mathbf{A}$ are identical or if $\mathbf{A}$ has a row or a column of zeroes, then $\det\mathbf{A} = 0$.
3. If the matrix $\mathbf{B}$ is obtained by multiplying a *single* row or a single column of $\mathbf{A}$ by a number α, then

   $$\det\mathbf{B} = \alpha\det\mathbf{A}.$$

   If *all* n rows (or all columns) of $\mathbf{A}$ are multiplied by α to obtain $\mathbf{B}$, then

   $$\det\mathbf{B} = \alpha^n\det\mathbf{A}.$$

4. If $\mathbf{B}$ is obtained by interchanging two rows of $\mathbf{A}$, then
$$\det\mathbf{B} = -\det\mathbf{A}.$$

5. If $\mathbf{B}$ is obtained by adding a multiple of one row (or column) of $\mathbf{A}$ to another, then
$$\det\mathbf{B} = \det\mathbf{A}.$$

Example: use row reduction to compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{bmatrix} =$$

Solution:

Interchange R_2 and R_3: $$\det\mathbf{A} = -\begin{vmatrix} 2 & 3 & 3 & 1 \\ 2 & -1 & -1 & -3 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix} =$$

(note that the determinant changes sign, by property 4 above)

$R_2 - R_1 \longrightarrow R_2$ $$= -\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix} =$$

(determinant does not change)

$R_4 + R_3 \longrightarrow R_4$ $$= -\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & 4 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{vmatrix} =$$

(determinant does not change)

$R_3 + R_2 \longrightarrow R_3$ $$= -\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & -1 \end{vmatrix} =$$

(determinant does not change, and we get an upper triangular matrix)

Compute the determinant of the upper triangular matrix: $$= -2 \cdot (-4) \cdot (-1) \cdot (-1) = 8$$

4 Homework problems

1. For each of the following matrices, find the inverse or show that it does not exist. In the latter case, check by calculating the determinant.

$$\text{a)}\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \text{b)}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{c)}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \qquad \text{d)}\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix}$$

2. Use the method of row reduction to evaluate the following determinants:

$$\text{a)}\begin{vmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & -2 \end{vmatrix} \qquad \text{b)}\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ -2 & 3 & -2 & 3 \\ 0 & -3 & 3 & 3 \end{vmatrix}$$