MATRICES

Matrices are rectangular arrays of real or complex numbers. With them, we define arithmetic operations that are generalizations of those for real and complex numbers. The general form a matrix of order \( m \times n \) is

\[
A = \begin{bmatrix}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
a_{m,1} & \cdots & a_{m,n}
\end{bmatrix}
\]

We say it has order \( m \times n \). Matrices that consist of a single column are called ‘column vectors’, and those consisting of a single row are called ‘row vectors’. In both cases, they will have properties identical to the geometric vectors studied earlier in multivariable calculus. I assume that all of you have seen this material previously in a course named linear algebra, matrix algebra, or something similar. §8.3 in the text is intended as both a quick introduction and review of this material.
MATRIX ADDITION

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be matrices of order $m \times n$. Then

$$C = A + B$$

is another matrix of order $m \times n$, with

$$c_{i,j} = a_{i,j} + b_{i,j}$$

**EXAMPLE.**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 5 \\ 6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
MATRIX MULTIPLICATION

Let \( A = \begin{bmatrix} a_{i,j} \end{bmatrix} \) have order \( m \times n \) and \( B = \begin{bmatrix} b_{i,j} \end{bmatrix} \) have order \( n \times p \). Then

\[
C = AB
\]

is a matrix of order \( m \times p \) and

\[
c_{i,j} = A_{i,*}B_{*,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j}
\]

or equivalently

\[

c_{i,j} = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,n} \end{bmatrix} \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{n,j} \end{bmatrix} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j}
\]

**Multiplication by a constant:**

\[
c \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} ca_{1,1} & \cdots & ca_{1,n} \\ \vdots & \ddots & \vdots \\ ca_{m,1} & \cdots & ca_{m,n} \end{bmatrix}
\]
EXAMPLES

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix} =
\begin{bmatrix}
22 & 28 \\
49 & 64
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix} =
\begin{bmatrix}
9 & 12 & 15 \\
19 & 26 & 33 \\
29 & 40 & 51
\end{bmatrix}
\]

\[
\begin{bmatrix}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
a_{n,1} & \cdots & a_{n,n}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
a_{1,1}x_1 + \cdots + a_{1,n}x_n \\
\vdots \\
a_{n,1}x_1 + \cdots + a_{n,n}x_n
\end{bmatrix}
\]

Thus we write the linear system

\[
a_{1,1}x_1 + \cdots + a_{1,n}x_n = b_1 \\
\vdots \\
a_{n,1}x_1 + \cdots + a_{n,n}x_n = b_n
\]

as

\[
Ax = b
\]
THE ZERO MATRIX 0

Define the zero matrix of order $m \times n$ as the matrix of that order having all zero entries. It is sometimes written as $0_{m \times n}$, but more commonly as simply 0. Then for any matrix $A$ of order $m \times n$,

$$A + 0 = 0 + A = A$$

The zero matrix $0_{m \times n}$ acts in the same role as does the number zero when doing arithmetic with real and complex numbers.

We denote by $-A$ the solution of the equation

$$A + B = 0$$

It is the matrix obtained by taking the negative of all of the entries in $A$. For example,

$$- \begin{bmatrix}
  a_{1,1} & a_{1,2} \\
  a_{2,1} & a_{2,2}
\end{bmatrix} = \begin{bmatrix}
  -a_{1,1} & -a_{1,2} \\
  -a_{2,1} & -a_{2,2}
\end{bmatrix}$$
THE IDENTITY MATRIX $I$

For a given integer $n \geq 1$, Define $I_n$ to be the matrix of order $n \times n$ with 1’s in all diagonal positions and zeros elsewhere:

$$I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \end{bmatrix}$$

More commonly it is denoted by simply $I$.

Let $A$ be a matrix of order $m \times n$. Then

$$AI_n = A, \quad I_mA = A$$

The identity matrix $I$ acts in the same role as does the number 1 when doing arithmetic with real and complex numbers.
THE MATRIX INVERSE

Let $A$ be a matrix of order $n \times n$ for some $n \geq 1$. We say a matrix $B$ is an inverse for $A$ if

$$AB = BA = I$$

It can be shown that if an inverse exists for $A$, then it is unique (see the argument on page 233).

EXAMPLES. If $ad - bc \neq 0$, then

$$
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
    d & -b \\
    -c & a
\end{bmatrix}
$$

$$
\begin{bmatrix}
    1 & 2 \\
    2 & 2
\end{bmatrix}^{-1} = \begin{bmatrix}
    -1 & 1 \\
    1 & -\frac{1}{2}
\end{bmatrix}
$$

$$
\begin{bmatrix}
    1 & \frac{1}{2} & \frac{1}{3} \\
    \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
    \frac{1}{3} & \frac{1}{4} & \frac{1}{5}
\end{bmatrix}^{-1} = \begin{bmatrix}
    9 & -36 & 30 \\
    -36 & 192 & -180 \\
    30 & -180 & 180
\end{bmatrix}
$$
Recall the earlier theorem on the solution of linear systems $Ax = b$ with $A$ a square matrix.

**Theorem.** The following are equivalent statements.

1. For each $b$, there is exactly one solution $x$.

2. For each $b$, there is a solution $x$.

3. The homogeneous system $Ax = 0$ has only the solution $x = 0$.

4. $\det(A) \neq 0$.

5. $A^{-1}$ exists.
EXAMPLE

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix} = 0
\]

Therefore, the linear system

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix}
\]

is not always solvable, the coefficient matrix does not have an inverse, and the homogeneous system \( Ax = 0 \) has a solution other than the zero vector, namely

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
-2 \\
1 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
The arithmetic properties of matrix addition and multiplication are listed on page 232, and some of them require some work to show. For example, consider showing the distributive law for matrix multiplication,

\[(AB)C = A(BC)\]

with \(A, B, C\) matrices of respective orders \(m \times n, n \times p,\) and \(p \times q\), respectively. Writing this out, we want to show

\[
\sum_{k=1}^{p} (AB)_{i,k} C_{k,l} = \sum_{j=1}^{n} A_{i,j} (BC)_{j,l}
\]

for \(1 \leq i \leq m, 1 \leq l \leq q\).

With new situations, we often use notation to suggest what should be true. But this is done only after deciding what actually is true. You should read carefully the properties given in the text on page 232.
PARTITIONED MATRICES

Matrices can be built up from smaller matrices; or conversely, we can decompose a large matrix into a matrix of smaller matrices. For example, consider

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
2 & 1 & 1 \\
0 & -1 & 5
\end{bmatrix} = \begin{bmatrix}
B & c \\
d & e
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix} \quad c = \begin{bmatrix}
0 \\
1
\end{bmatrix} \quad d = \begin{bmatrix}
0 & -1
\end{bmatrix} \quad e = 5
\]

Matlab allows you to build up larger matrices out of smaller matrices in exactly this manner; and smaller matrices can be defined as portions of larger matrices. We will often write an \( n \times n \) square matrix in terms of its columns:

\[
A = [A_{*,1}, \ldots, A_{*,n}]
\]

For the \( n \times n \) identity matrix \( I \), we write

\[
I = [e_1, \ldots, e_n]
\]

with \( e_j \) denoting a column vector with a 1 in position \( j \) and zeros elsewhere.
ARITHMETIC OF PARTITIONED MATRICES

As with matrices, we can do addition and multiplication with partitioned matrices provided the individual constituent parts have the proper orders.

For example, let $A, B, C, D$ be $n \times n$ matrices. Then

$$
\begin{bmatrix}
I & A \\
B & I
\end{bmatrix}
\begin{bmatrix}
I & C \\
D & I
\end{bmatrix} =
\begin{bmatrix}
I + AD & C + A \\
B + D & I + BC
\end{bmatrix}
$$

Let $A$ be $n \times n$ and $x$ be a column vector of length $n$. Then

$$
Ax = \begin{bmatrix}
A_{*,1}, \ldots, A_{*,n}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} = x_1A_{*,1} + \cdots + x_nA_{*,n}
$$

Compare this to

$$
\begin{bmatrix}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
a_{n,1} & \cdots & a_{n,n}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
a_{1,1}x_1 + \cdots + a_{1,n}x_n \\
\vdots \\
a_{n,1}x_1 + \cdots + a_{n,n}x_n
\end{bmatrix}
$$
CALCULATING THE MATRIX INVERSE

Consider finding the inverse of a $3 \times 3$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = [A_*,1, A_*,2, A_*,3]$$

We want to find a matrix

$$X = [X_*,1, X_*,2, X_*,3]$$

for which

$$AX = I$$

$$A [X_*,1, X_*,2, X_*,3] = [e_1, e_2, e_3]$$

$$[AX_*,1, AX_*,2, AX_*,3] = [e_1, e_2, e_3]$$

This means we want to solve

$$AX_*,1 = e_1, \quad AX_*,2 = e_2, \quad AX_*,3 = e_3$$

We want to solve three linear systems, all with the same matrix of coefficients $A$. 
In augmented matrix notation, we want to work with

\[
[A \mid I]
\]

\[
\begin{bmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} & 1 & 0 & 0 \\
  a_{2,1} & a_{2,2} & a_{2,3} & 0 & 1 & 0 \\
  a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 1 \\
\end{bmatrix}
\]

Then we proceed as before with a single linear system, only now we have three right hand sides. We first introduce zeros into positions 2 and 3 of column 1; and then we introduce zero into position 3 of column 2. Following that, we will need to solve three upper triangular linear systems by back substitution. See the numerical example on page 231.
MATRIX INVERSE EXAMPLE

\[ A = \begin{bmatrix}
1 & 1 & -2 \\
1 & 1 & 1 \\
1 & -1 & 0
\end{bmatrix} \]

\[
\begin{bmatrix}
1 & 1 & -2 & | & 1 & 0 & 0 \\
1 & 1 & 1 & | & 0 & 1 & 0 \\
1 & -1 & 0 & | & 0 & 0 & 1
\end{bmatrix}
\]

\[ m_{2,1} = 1 \quad \text{and} \quad m_{3,1} = 1 \]

\[
\begin{bmatrix}
1 & 1 & -2 & | & 1 & 0 & 0 \\
0 & 0 & 3 & | & -1 & 1 & 0 \\
0 & -2 & 2 & | & -1 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & -2 & | & 1 & 0 & 0 \\
0 & -2 & 2 & | & -1 & 0 & 1 \\
0 & 0 & 3 & | & -1 & 1 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 1 & -2 & 1 & 0 & 0 \\
0 & -2 & 2 & -1 & 0 & 1 \\
0 & 0 & 3 & -1 & 1 & 0
\end{bmatrix}
\]

Then by using back substitution to solve for each column of the inverse, we obtain

\[
A^{-1} = \begin{bmatrix}
\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\
-\frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix}
\]
MATLAB MATRIX OPERATIONS

In Matlab, the command

$$\text{inv}(A)$$

will calculate the inverse of $A$. To solve the linear system $Ax = b$ in Matlab, use

$$x = A \backslash b$$

To obtain the $LU$-factorization of a matrix, including the use of partial pivoting, use the Matlab command $lu$. In particular,

$$[L, U, P] = \text{lu}(X)$$

returns the lower triangular matrix $L$, upper triangular matrix $U$, and permutation matrix $P$ so that

$$PX = LU$$