ROOTFINDING

We want to find the numbers $x$ for which $f(x) = 0$, with $f$ a given function. We denote such roots or zeroes by the Greek letter $\alpha$. Rootfinding problems occur in many contexts. Sometimes they are a direct formulation of some physical situation; but more often, they are an intermediate step in solving a much larger problem.

An example with annuities. Suppose you are paying into an account an amount of $P_{in}$ per period of time, for $N_{in}$ periods of time. The amount you are deposited is compounded at at interest rate of $r$ per period of time. Then at the beginning of period $N_{in} + 1$, you will withdraw an amount of $P_{out}$ per time period, for $N_{out}$ periods. In order that the amount you withdraw balance that which has been deposited including interest, what is the needed interest rate? The equation is

$$P_{in} \left[ (1 + r)^{N_{in}} - 1 \right] = P_{out} \left[ 1 - (1 + r)^{-N_{out}} \right]$$

We assume the interest rate $r$ holds over all $N_{in} + N_{out}$ periods.
As a particular case, suppose you are paying in $P_{in} = \$1,000$ each month for 40 years. Then you wish to withdraw $P_{out} = \$5,000$ per month for 20 years. What interest rate do you need? If the interest rate is $R$ per year, compounded monthly, then $r = R/12$. Also, $N_{in} = 40 \cdot 12 = 480$ and $N_{out} = 20 \cdot 12 = 240$. Thus we wish to solve

$$1000 \left[ \left( 1 + \frac{R}{12} \right)^{480} - 1 \right] = 5000 \left[ 1 - \left( 1 + \frac{R}{12} \right)^{-240} \right]$$

What is the needed yearly interest rate $R$? The answer is 2.92%. How did I obtain this answer?

This example also shows the power of compound interest.
THE BISECTION METHOD

Most methods for solving $f(x) = 0$ are iterative methods. We begin with the simplest of such methods, one which most people use at some time.

We assume we are given a function $f(x)$; and in addition, we assume we have an interval $[a, b]$ containing the root, on which the function is continuous. We also assume we are given an error tolerance $\varepsilon > 0$, and we want an approximate root $\tilde{\alpha}$ in $[a, b]$ for which

$$|\alpha - \tilde{\alpha}| \leq \varepsilon$$

We further assume the function $f(x)$ changes sign on $[a, b]$, with

$$f(a) f(b) < 0$$
Algorithm Bisect($f, a, b, \varepsilon$).  

**Step 1:** Define

$$c = \frac{1}{2} (a + b)$$

**Step 2:** If $b - c \leq \varepsilon$, accept $c$ as our root, and then stop.

**Step 3:** If $b - c > \varepsilon$, then check compare the sign of $f(c)$ to that of $f(a)$ and $f(b)$. If

$$\text{sign}(f(b)) \cdot \text{sign}(f(c)) \leq 0$$

then replace $a$ with $c$; and otherwise, replace $b$ with $c$. Return to Step 1.

Denote the initial interval by $[a_1, b_1]$, and denote each successive interval by $[a_j, b_j]$. Let $c_j$ denote the center of $[a_j, b_j]$. Then

$$|\alpha - c_j| \leq b_j - c_j = c_j - a_j = \frac{1}{2} (b_j - a_j)$$

Since each interval decreases by half from the preceding one, we have by induction,

$$|\alpha - c_n| \leq \left( \frac{1}{2} \right)^n (b_1 - a_1)$$
An example from the matlab program ‘bisect.m’ is given in an accompanying file. It is for the function.

\[ f(r) = P_{in} \left( (1 + r)^{N_{in}} - 1 \right) - P_{out} \left[ 1 - (1 + r)^{-N_{out}} \right] \]

Checking, we see that \( f(0) = 0 \). Therefore, with a graph of \( y = f(r) \) on \([0, 1]\), we see that \( f(x) < 0 \) if we choose \( x \) very small, say \( x = .001 \). Also \( f(1) > 0 \). Thus we choose \([a, b] = [.001, 1]\). Using \( \varepsilon = .000001 \) yields the answer

\[ \tilde{\alpha} = .02918243 \]

with an error bound of

\[ |\alpha - c_n| \leq 9.53 \times 10^{-7} \]

for \( n = 20 \) iterates. We could also have calculated this error bound from

\[ \frac{1}{2^{20}} (1 - .001) = 9.53 \times 10^{-7} \]
Suppose we are given the initial interval \([a, b] = [1.6, 4.5]\) with \(\varepsilon = .00005\). How large need \(n\) be in order to have

\[|\alpha - c_n| \leq \varepsilon\]

Recall that

\[|\alpha - c_n| \leq \left(\frac{1}{2}\right)^n (b - a)\]

Then ensure the error bound is true by requiring and solving

\[\left(\frac{1}{2}\right)^n (b - a) \leq \varepsilon\]

\[\left(\frac{1}{2}\right)^n (4.5 - 1.6) \leq .00005\]

Dividing and solving for \(n\), we have

\[n \geq \log \left(\frac{2.9}{.00005}\right) = 15.82\]

Therefore, we need to take \(n = 16\) iterates.
ADVANTAGES AND DISADVANTAGES

Advantages: 1. It always converges.
2. You have a guaranteed error bound, and it decreases with each successive iteration.
3. You have a guaranteed rate of convergence. The error bound decreases by $\frac{1}{2}$ with each iteration.

Disadvantages: 1. It is relatively slow when compared with other rootfinding methods we will study, especially when the function $f(x)$ has several continuous derivatives about the root $\alpha$.
2. The algorithm has no check to see whether the $\varepsilon$ is too small for the computer arithmetic being used. [This is easily fixed by reference to the unit round of the computer arithmetic.]

We also assume the function $f(x)$ is continuous on the given interval $[a, b]$; but there is no way to confirm this.