COMPUTING ANOMALIES

These examples are meant to help motivate the study of machine arithmetic.

1. **Calculator example:** Use an HP-15C calculator, which contains 10 digits in its display. Let

   \[ x_1 = x_2 = x_3 = 98765 \]

   There are keys on the calculator for the mean

   \[ \bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j \]

   and the standard deviation \( s \) where

   \[ s^2 = \frac{1}{n - 1} \sum_{j=1}^{n} (\bar{x} - x_j)^2 \]

   In our case, what should these equal? In fact, the calculator gives

   \[ \bar{x} = 98765 \quad s \approx 1.58 \]

   Why?
2. A Fortran program example: Consider two programs run on a now extinct computer.

Program A:

\[
A = 1.0 + 2.0 \times (-23) \\
B = A - 1.0 \\
PRINT *, A, B \\
END
\]

Output: 1.0 \quad 1.19E - 7 \quad (\doteq 2^{-23})

Program B:

\[
A = 1.0 + 2.0 \times (-23) \\
SILLY = 0.0 \\
B = A - 1.0 \\
PRINT *, A, B \\
END
\]

Output: 1.0 \quad 0.0

Why the change, since presumably the variable \texttt{SILLY} does not have any connection to \texttt{B}.
BINARY NUMBERS

A binary integer $x$ is a finite sequence of the digits 0 and 1, which we write symbolically as

$$x = (a_m a_{m-1} \cdots a_2 a_1 a_0)_2$$

where I insert the parentheses with subscript $(\cdot)_2$ in order to make clear that the number is binary. The above has the decimal equivalent

$$x = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0$$

For example, the binary integer $x = (111 \cdots 1)_2$ with $m$ ones has the value

$$x = 2^{m-1} + \cdots + 2^1 + 1 = 2^m - 1$$
Conversion of decimal integers to binary integers: Given a decimal integer \( x \) we write

\[
x = (a_m a_{m-1} \cdots a_2 a_1 a_0)_2 \\
= a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0
\]

Divide \( x \) by 2, calling the quotient \( x_1 \). The remainder is \( a_0 \), and

\[
x_1 = a_m 2^{m-1} + a_{m-1} 2^{m-2} + \cdots + a_1 2^0
\]

Continue the process. Divide \( x_1 \) by 2, calling the quotient \( x_2 \). The remainder is \( a_1 \), and

\[
x_2 = a_m 2^{m-2} + a_{m-1} 2^{m-3} + \cdots + a_2 2^0
\]

After a finite number of such steps, we will obtain all of the coefficients \( a_i \), and the final quotient will be zero.

Try this with a few decimal integers.
A binary fraction $x$ is a sequence (possibly infinite) of the digits 0 and 1:

\[
x = (a_1a_2a_3\cdots a_m\cdots)_2 = a_12^{-1} + a_22^{-2} + a_32^{-3} + \cdots
\]

For example,

\[
(.0101010101010\cdots)_2 = 2^{-2} + 2^{-4} + 2^{-6} + \cdots
\]

which sums to the fraction $1/3$.

Also,

\[
(.11001100110011\cdots)_2 = 2^{-1} + 2^{-2} + 2^{-5} + 2^{-6} + \cdots
\]

and this sums to the decimal fraction $0.8 = \frac{8}{10}$. 
Conversion of decimal fractions to binary fractions: In

\[ x_1 = (a_1a_2a_3 \cdots a_m \cdots)_2 \]
\[ = a_12^{-1} + a_22^{-2} + a_32^{-3} + \cdots \]

we multiply by 2. The integer part will be \( a_1 \); and after it is removed we have the binary fraction

\[ x_2 = (a_2a_3 \cdots a_m \cdots)_2 \]
\[ = a_22^{-1} + a_32^{-2} + a_42^{-3} + \cdots \]

Again multiply by 2, obtaining \( a_2 \) as the integer part of \( 2x_2 \). After removing \( a_2 \), let \( x_3 \) denote the remaining number. Continue this process as far as needed.

For example, with \( x = \frac{1}{5} \), we have

\[ x_1 = .2; \quad 2x_1 = .4; \quad x_2 = .4 \text{ and } a_1 = 0 \]
\[ 2x_2 = .8; \quad x_3 = .8 \text{ and } a_2 = 0 \]
\[ 2x_3 = 1.6; \quad x_4 = .6 \text{ and } a_2 = 1 \]

Continue this to get the pattern

\[ (.2)_{10} = (.00110011001100 \cdots)_2 \]