NORMS

Norms are a measure of the “size” of objects. We introduce norms to measure the size of vectors and matrices.

Let $\mathcal{V}$ be a vector space. A norm on $\mathcal{V}$ is a non-negative real-valued function $N(x)$ defined on $\mathcal{V}$ and satisfying the following properties:

N1. $N(x) = 0$ if and only if $x = 0$.
N2. $N(\alpha x) = |\alpha| N(x)$, for all $x \in \mathcal{V}$ and all scalars $\alpha$.
N3. $N(x + y) \leq N(x) + N(y)$, for all $x, y \in \mathcal{V}$.

Using $N$, we define the distance between vectors $x$ and $y$ to be $N(x - y)$. From N3, we have

$$N(x - z) \leq N(x - y) + N(y - z), \quad x, y, z \in \mathcal{V}$$

We can also prove the reverse triangle inequality:

$$|N(x) - N(y)| \leq N(x - y), \quad x, y \in \mathcal{V}$$

More commonly, we write $\|x\| = N(x)$. 
EXAMPLES

Let $p$ be any real number satisfying $1 \leq p < \infty$. For $x \in \mathbb{R}^n$ or $\mathbb{C}^n$, introduce the $p$-norm:

$$
\|x\|_p = \left[ \sum_{j=1}^{n} |x_j|^p \right]^{\frac{1}{p}}
$$

This is called the $p$-norm. For $p = 1$, showing N1-N3 is relatively straightforward. For $p = 2$, we have the Euclidean norm introduced earlier and associated with an inner product.

Define the maximum norm by

$$
\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|
$$

It can be shown that

$$
\lim_{p \to \infty} \left[ \sum_{j=1}^{n} |x_j|^p \right]^{\frac{1}{p}} = \max_{i=1,\ldots,n} |x_i|, \quad x \in \mathbb{C}^n
$$

which is the justification for the notation $\|x\|_\infty$. 
Proof that

\[ \|x\|_1 = \sum_{i=1}^{n} |x_i| \]

is a norm. Our vector space is \( \mathcal{V} = \mathbb{R}^n \) or \( \mathbb{C}^n \).

N1. Let \( x \in \mathcal{V} \). Clearly \( \|x\|_1 \geq 0 \). Also \( \|x\|_1 = 0 \) if and only if all components \( x_i = 0 \), or equivalently, \( x = 0 \).

N2. Let \( x \in \mathcal{V} \), and let \( \alpha \) be a scalar. Then

\[ \alpha x = [\alpha x_1, ..., \alpha x_n]^T \]

and

\[ \|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha| \|x\|_1 \]

N3. Let \( x, y \in \mathcal{V} \). Then

\[ x + y = [x_1 + y_1, ..., x_n + y_n]^T \]

\[ \|x + y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\|_1 + \|y\|_1 \]
The quantity
\[ \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad x \in \mathbb{R}^n \text{ or } \mathbb{C}^n \]

can be shown to be a norm by similar arguments. This particular norm was introduced in Chapter 1; and it is the discrete analogue to the function norm
\[ \|f\|_\infty = \max_{a \leq t \leq b} |f(t)|, \quad f \in C[a, b] \]

The quantity
\[ \|x\|_2 = \left[ \sum_{j=1}^{n} |x_j|^2 \right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^n \text{ or } \mathbb{C}^n \]
is the Euclidean norm studied previously in Section 7.2, where we showed it satisfied the triangle inequality. It is the discrete analogue of the function norm
\[ \|f\|_2 = \left[ \int_{a}^{b} |f(t)|^2 \, dt \right]^{\frac{1}{2}}, \quad f \in C[a, b] \]
CONVERGENCE IN $\mathbb{R}^n$ AND $\mathbb{C}^n$

We say a sequence of vectors $\{x^{(1)}, ..., x^{(m)}, ...\}$ in $\mathcal{V} = \mathbb{R}^n$ (or $\mathbb{C}^n$) converges to a vector $x \in \mathcal{V}$ if and only if

$$\lim_{m \to \infty} \|x - x^{(m)}\| = 0$$

What norm am I using? It turns out to not matter. As a lead-in to this, note that

$$\max_{1 \leq i \leq n} |x_i| \leq \sum_{i=1}^{n} |x_i| \leq n \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_{\infty} \leq \|x\|_1 \leq n \|x\|_{\infty}, \quad x \in \mathcal{V}$$

Therefore

$$\|x - x^{(m)}\|_{\infty} \leq \|x - x^{(m)}\|_1 \leq n \|x - x^{(m)}\|_{\infty}, \quad x \in \mathcal{V}$$

As $m \to \infty$, convergence with the norm $\|\cdot\|_1$ is equivalent to convergence with the norm $\|\cdot\|_{\infty}$. 
EQUIVALENCE OF NORMS

Let $N(x)$ and $M(x)$ denote two norms on the vector space $\mathcal{V} = \mathbb{R}^n$ or $\mathbb{C}^n$. Then there are positive constants $c_1$ and $c_2$ for which

$$c_1 M(x) \leq N(x) \leq c_2 M(x), \quad x \in \mathcal{V}$$

Note that this says convergence is equivalent in all norms, as

$$c_1 M(x - x^{(m)}) \leq N(x - x^{(m)}) \leq c_2 M(x - x^{(m)})$$

Proof. How do we shown this result. A proof is given in the text if the following is too brief. We begin by letting $M(x) = \|x\|_\infty$, as that will be sufficient to show the general case. Then we need to show

$$c_1 \|x\|_\infty \leq N(x) \leq c_2 \|x\|_\infty, \quad x \in \mathcal{V}$$
Divide by $\|x\|_\infty$ to obtain

$$c_1 \leq N \left( \frac{x}{\|x\|_\infty} \right) \leq c_2, \quad x \in \mathcal{V}$$

or equivalently, show

$$c_1 \leq N(z) \leq c_2, \quad \text{for all } z \in \mathcal{V}, \quad \|z\|_\infty = 1$$

Boundedness of $N(x)$. For general $x$, write

$$x = [x_1, \ldots, x_n]^T = \sum_{j=1}^{n} x_j e^{(j)}$$

with $e^{(j)}$ the standard unit vectors. Then

$$N(x) = N \left( \sum_{j=1}^{n} x_j e^{(j)} \right) \leq \sum_{j=1}^{n} |x_j| N \left( e^{(j)} \right) \leq c_2 \|x\|_\infty$$

with

$$c_2 = \sum_{j=1}^{n} N \left( e^{(j)} \right)$$

Next we need to show

$$c_1 \leq N(z), \quad \text{for all } z \in \mathcal{V}, \quad \|z\|_\infty = 1$$

for some $c_1 > 0$. 
The above boundedness result also shows that $N$ is a continuous function. In particular,

$$|N(x) - N(y)| \leq N(x - y) \leq c_2 \|x - y\|_{\infty}$$

Thus $N(x)$ is a continuous function of $x$.

Introduce the set

$$S = \{z \in \mathcal{V} \mid \|z\|_{\infty} = 1\}$$

This is called the *unit sphere* relative to the norm $\|\cdot\|_{\infty}$. It is a closed and bounded set in $\mathcal{V}$, relative to $\|\cdot\|_{\infty}$; and $N(z)$ is a continuous real-valued function as $z$ varies over this set. Therefore, the maximum and minimum of $N(z)$ over $S$ occurs at points $z^*$ and $z_*$ in $S$:

$$c_2 \equiv \max_{z \in S} N(z) = N(z^*)$$

$$c_1 \equiv \min_{z \in S} N(z) = N(z_*)$$

Since points in $S$ are nonzero, we have $c_1 \neq 0$. This completes the proof.
MATRIX NORMS

Notice that sets of matrices can be considered as a vector space. For example, consider all $2 \times 2$ matrices with real entries:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

Then this set of matrices is a vector space $\mathcal{V}$:

$$\alpha \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} \alpha a_{1,1} & \alpha a_{1,2} \\ \alpha a_{2,1} & \alpha a_{2,2} \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} \end{bmatrix}$$

The dimension of this vector space is 4.

The set of matrices of order $m \times n$ form a vector space of dimension $mn$. Thus every matrix norm must first be a vector norm.
Let $\mathcal{M}$ denote the set of square matrices of order $n \times n$. Then a matrix norm on $\mathcal{M}$ is a non-negative real-value function $\|A\|$ for which

N1. $\|A\| = 0$ if and only if $A = 0$.
N2. $\|\alpha A\| = |\alpha| \|A\|$, for all scalars $\alpha$ and all $A \in \mathcal{M}$.
N3. $\|A + B\| \leq \|A\| + \|B\|$, for all $A, B \in \mathcal{M}$.

In addition, matrices have a sense of multiplication, involving both vectors and other matrices. We would like to have this recognized in defining a sense of size of a matrix. In particular, we require a matrix norm to also satisfy the following.
N4. $\|AB\| \leq \|A\| \|B\|$, $A, B \in \mathcal{M}$

For $\mathcal{V} = \mathbb{R}^n$ or $\mathbb{C}^n$, we will have some vector norm $\|x\|_v$ we are using. Then we require a “compatibility” between the matrix norm and the vector norm:

N5.

$$\|Ax\|_v \leq \|A\| \|x\|_v, \quad x \in \mathcal{V}, \quad A \in \mathcal{M}$$

In defining a matrix norm, we usually begin with the vector norm and then discover how to define the matrix norm so as to maintain compatibility with the vector norm.
Example: Consider the vector norm

\[ \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad x \in \mathbb{R}^n \]

Then

\[ Ax = \left[ \sum_{j=1}^{n} a_{1,j} x_j, \ldots, \sum_{j=1}^{n} a_{n,j} x_j \right]^T \]

\[ \|Ax\|_\infty = \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{i,j} x_j \right| \]

We want to have this be bounded by \( \|A\| \|x\|_\infty \) for some choice of \( \|A\| \). Then

\[ \|Ax\|_\infty \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{i,j}| |x_j| \]

\[ \leq \left[ \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{i,j}| \right] \|x\|_\infty \]
Define
\[ \|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{i,j}| \]

Is this a norm? We know the property N5 is satisfied since we constructed it that way. We can also show this definition satisfies properties N1-N4.

There is a more general way to proceed in getting the norm, and the above is an example of this process. When given a vector norm $\|\cdot\|_v$, we want to have the matrix norm to satisfy
\[ \|Ax\|_v \leq \|A\| \|x\|_v, \quad x \in \mathcal{V}, \quad A \in \mathcal{M} \]

To accomplish this, we define
\[ \|A\| = \sup_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v} \]

This is called the **operator matrix norm** associated with the given vector norm $\|\cdot\|_v$. 
SOME OPERATOR NORMS

Introduce

$$\sigma(A) = \{ \lambda \mid \lambda \text{ an eigenvalue of } A \}$$

$$r_\sigma(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

The set $\sigma(A)$ is called the spectrum of $A$; and the number $r_\sigma(A)$ is called the spectral radius of $A$.

**Vector norm**  

<table>
<thead>
<tr>
<th>Vector norm</th>
<th>Operator matrix norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|x|_\infty$</td>
<td>$|A|<em>\infty \equiv \max</em>{1 \leq i \leq n} \sum_{j=1}^{n}</td>
</tr>
<tr>
<td>$|x|_1$</td>
<td>$|A|<em>1 \equiv \max</em>{1 \leq j \leq n} \sum_{i=1}^{n}</td>
</tr>
<tr>
<td>$|x|_2$</td>
<td>$|A|<em>2 \equiv [r</em>\sigma(AA^*)]^{\frac{1}{2}} = [r_\sigma(A^*A)]^{\frac{1}{2}}$</td>
</tr>
</tbody>
</table>
\(\|A\|_\infty\) is called the row norm; and \(\|A\|_1\) is called the column norm. The quantity \(\|A\|_2\) is often hard to compute, but it is bounded by the Frobenius norm:

\[
F(A) \equiv \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|^2 \right]^{1/2}
\]

Properties: (a) For the identity matrix \(A = I\), we have

\[
\|I\| = \sup_{x \neq 0} \frac{\|Ix\|_v}{\|x\|_v} = \sup_{x \neq 0} \frac{\|x\|_v}{\|x\|_v} = 1
\]

(b) Let \(\lambda\) be an eigenvalue of \(A\), with \(x \neq 0\) an associated eigenvector. Then

\[
|\lambda| \|x\|_v = \|\lambda x\|_v = \|Ax\|_v \leq \|A\| \|x\|_v
\]

\[
|\lambda| \leq \|A\|
\]

This proves

\[
r_\sigma(A) \leq \|A\|
\]

for any operator matrix norm.
(c) Let $A$ be a real symmetric matrix or a complex Hermitian matrix. Then

$$A^* A = AA^* = A^2$$

For its eigenvalues, let $u_1, \ldots, u_n$ be an orthogonal set of eigenvectors of $A$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$A^2 u_i = A(Au_i) = A(\lambda_i u_i) = \lambda_i^2 u_i$$

Thus when $A$ is Hermitian,

$$\|A\|_2 = [r_\sigma(AA^*)]^{\frac{1}{2}} = [r_\sigma(A^2)]^{\frac{1}{2}} = r_\sigma(A)$$

Thus $\|A\|_2$ may be easier to compute when $A$ is symmetric or Hermitian.

There are other properties that are explored in the assigned problems.
THEOREM

Let $\epsilon > 0$ be a given small number and let $A$ be a square matrix. Then there is a special vector norm $\| \cdot \|_v$ and an associated matrix operator matrix norm $\| \cdot \|_\epsilon$ for which

$$\|A\|_\epsilon \leq r_\sigma(A) + \epsilon$$

This not an easy theorem to prove, and I give a reference to another text for a proof. It shows that $r_\sigma(A)$ is close to some operator matrix norm for $A$, since then

$$r_\sigma(A) \leq \|A\|_\epsilon \leq r_\sigma(A) + \epsilon$$

This proves the result that $r_\sigma(A) < 1$ if and only if $\|A\| < 1$ for some operator matrix norm.