1. (35 pts.) Find the derivatives of the following functions (you need not simplify):

(a). $f(x) = \ln(x^3 + 2x) + e^{-x^2} - \frac{1}{x^3} + 3$

$$f'(x) = \frac{3x^2+2}{x^3+2x} - (2x)e^{-x^2} + \frac{3}{x^4}$$

(b) $f(x) = 2x^e + e^{2x}$.

$$f'(x) = 2e\,x^{e-1} + 2e^{2x}.$$

(c) $f(x) = \frac{e^{1-x}}{x^2+x}$

$$f'(x) = \frac{-(x^2+x)e^{1-x} - (2x+1)e^{1-x}}{(x^2+x)^2}.$$

(d) $f(x) = (e^x - x)^7$

$$f'(x) = 7(e^x - x)^6 (e^x - 1)$$

(e). $f(x) = (\sqrt{x})\ln(x)$

$$f'(x) = \sqrt{x} \cdot \frac{1}{x} + \frac{1}{2\sqrt{x}} \ln x.$$

2. (8 pts.) A patient receives an injection of 1.5 milligrams of a drug, and the amount remaining in the bloodstream t hours later is $A(t) = 1.5e^{-0.08t}$. Find the instantaneous rate of change and the relative rate of change of the amount in the bloodstream at time t = 0.(immediately after the injection).

Inst. r.o.c. = $A'(0)$

$$A'(t) = -(.08)(1.5)e^{-.08t} = -.12\,e^{-.08t}$$

$$A'(0) = -.12 \ \frac{mg}{hr}$$

Relative r.o.c. $= \frac{A'(0)}{A(0)} = \frac{-.12}{1.5} = -\frac{.08}{hr}$ or -8% per hour.

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3. (15 pts) $f(x) = x^3 - 3x^2 + 3$. Make sign diagrams for the 1st and 2nd derivatives and sketch the graph of y = f(x) by hand, labeling all relative extrema and inflection points.

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x) = 0$ at $x = 0, x = 2$.

$f'>0$ $\quad$ $f'<0$ $\quad$ $f'>0$ $\quad$ $f'(x)$

$f'=0$ $\quad$ $f'=0$ $\quad \rightarrow x$

$x=0$ $\quad$ $x=2$

$f''(x) = 6x - 6 = 6(x-1)$

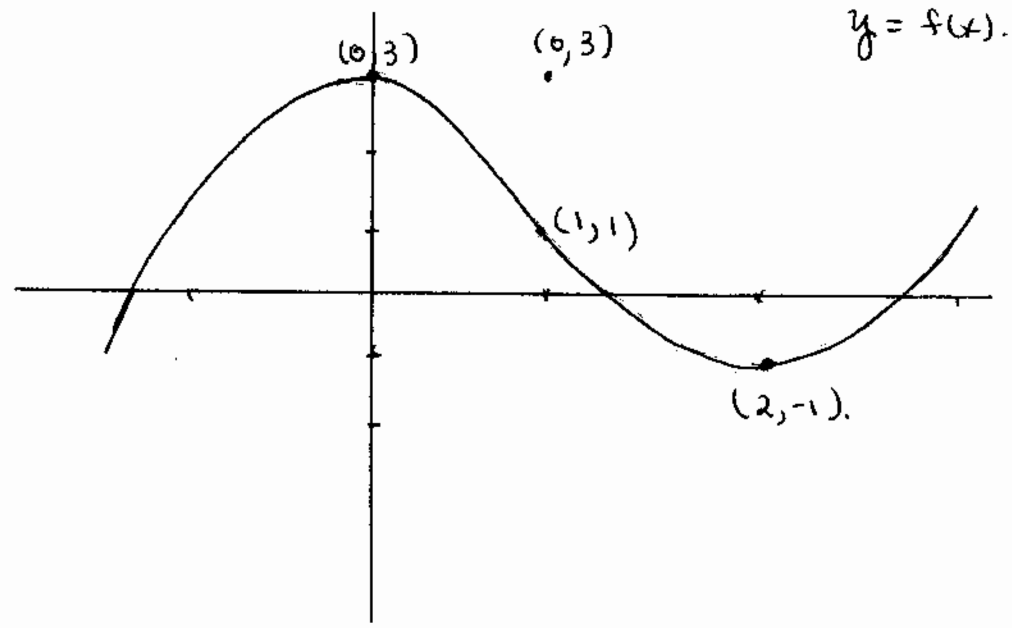
$f''(x) = 0$ at $x = 1$

$f''(x)$

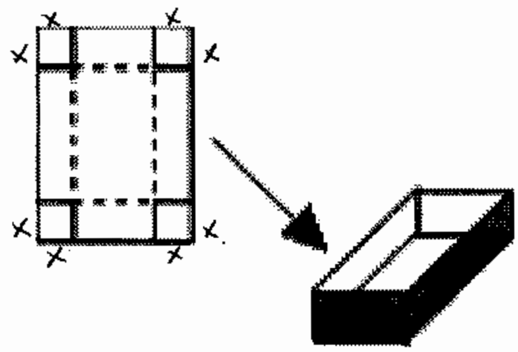
$f'<0$ $\quad$ $f'=0$ $\quad$ $f'>0$ $\quad \rightarrow x$

$x=1$

Relative max. at (0, 3); relative min. at (2, -1)
Inflection pt. at (1, 1)



4. (12 pts.) You have a square piece of cardboard 12 inches wide and 12 inches long that you wish to fold into a box. It occurs to you that you can cut an equal square from each corner of the cardboard, make a crease along each side and fold the sides up, as indicated in the diagram. How much should you cut from the corners to form the box with maximum volume? Volume = (length)(width)(height).



$V(x) = (12-2x)(12-2x)x, \quad 0<x<6$

$V(x) = (144-48x+4x^2)x$

$V(x) = 144x - 48x^2 + 4x^3$

$V'(x) = 144 - 96x + 12x^2 = 12(12-8x+x^2)$

$V'(x) = 12(x-2)(x-6)$

$V'(x) = 0$ at $x=2$, $\cancel{x=6}$.

$V''(x) = -96+24x = 24(x-4)$

$V''(2) < 0$

The maximum volume occurs when 2 inches are cut from each corner.

5. (10 pts) A demand function is given by D(p) = 200 - 5p. Find the elasticity of demand, E(p) at p = 10. Determine whether the demand is elastic, inelastic, or unitary at p=10. Should the price be raised or lowered to increase revenue?

$D'(p) = -5$

$E(p) = -\dfrac{p\,D'(p)}{D(p)} = \dfrac{5p}{200-5p}$

$E(10) = \dfrac{50}{150} = \dfrac{1}{3}$

$E(10) = \frac{1}{3} < 1$, so demand is inelastic at $p = 10$.

The price should be raised to increase revenue.

6 (12 pts.) A wine warehouse expects to sell 30,000 bottles of wine in a year. Each bottle costs $9, and there is a fixed charge of $200 per order. If it costs $3 to store a bottle for a year, how many bottles should be ordered at a time and how many orders should the warehouse place in a year to minimize inventory costs?

Let x be the lot size, or number of bottles ordered at a time

Storage Cost $= \left(\dfrac{x}{2}\right)3 = \dfrac{3x}{2}$

Reorder Cost $= (9x+200)\left(\dfrac{30000}{x}\right) = 270{,}000 + \dfrac{6{,}000{,}000}{x}$

Inventory Cost, $C(x) = \dfrac{3x}{2} + 270{,}000 + \dfrac{6{,}000{,}000}{x}$, $x > 0$

$C'(x) = \dfrac{3}{2} - \dfrac{6{,}000{,}000}{x^2}$

$C''(x) = \dfrac{12{,}000{,}000}{x^3}$

$C'(x) = 0$ for $x^2 = 4{,}000{,}000$ or $x = 2000$

$C''(2000) > 0$, so 2000 bottles should be ordered at a time to minimize inventory costs.

E

7. (8 pts.) $x^2y+xy^3-3x=8$. Find $\frac{dy}{dx}$ at x=4, y=1.

$$2xy+x^2\frac{dy}{dx}+y^3+3xy^2\frac{dy}{dx}-3=0$$

$$x^2\frac{dy}{dx}+3xy^2\frac{dy}{dx}=3-2xy-y^3$$

$$\frac{dy}{dx}(x^2+3xy^2)=3-2xy-y^3$$

$$\frac{dy}{dx}=\frac{3-2xy-y^3}{x^2+3xy^2}$$

$$\left.\frac{dy}{dx}\right|_{(4,1)}=\frac{3-8-1}{16+12}=-\frac{6}{28}=-\frac{3}{14}$$

(10 pts) Extra-Credit Problem: It costs a bicycle manufacturer \$70 to manufacture each bicycle, and the fixed costs are \$100 per day. The price function is p(x) = 270 –10x, where p is the price in dollars at which exactly x bicycles will be sold. Find the number of bicycles the manufacturer should produce and the price it should charge to maximize the profit.

$C(x)=70x+100$

$R(x)=xp(x)=x(270-10x)=270x-10x^2$

$P(x)=R(x)-C(x)=200x-10x^2-100$

$P'(x)=200-20x$

$P'(x)=0$ for $x=10$

$P''(x)=-20$

The manufacturer should produce 10 bicycles and sell them for \$170 to maximize the profit.