Supplement Part: A Similarity Theory of Approximate Deconvolution Models of Turbulence

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May 9, 2006

1 Introduction

We consider the Navier-Stokes equations in a periodic box in $\mathbb{R}^3$:

$$\begin{aligned} u_t + u\cdot\nabla u - \nu\triangle u + \nabla p &= f \quad \text{in } \Omega=(0,L)^3,\ t>0, \\ \nabla\cdot u &= 0 \quad \text{in } (0,L)^3, \end{aligned} \tag{1.1}$$

subject to periodic (with zero mean) conditions

$$\begin{aligned} u(x+Le_j,t) &= u(x,t) \quad j=1,2,3 \quad \text{and}, \\ \int_\Omega \phi dx &= 0 \ \text{ for } \phi = u,\, u_0,\, f,\, p. \end{aligned} \tag{1.2}$$

LES computes an approximation to local spatial averages of solutions to (1.1)-(1.2). Many averaging operators are used in LES. Herein we choose a differential filter, [Ger86], associated with a length-scale $\delta > 0$. (The case of other filters is summarized in section 5.1.) Given $\phi(x)$, $\overline{\phi(x)}$ is the unique L-periodic solution of

$$A_\delta\overline{\phi} := -\delta^2\,\triangle\overline{\phi} + \overline{\phi} = \phi\,,\ \text{in } \Omega.$$

Averaging the NSE (meaning: applying A_δ^{-1} to (1.1)) gives the exact space filtered NSE for $\overline{u}$

$$\begin{aligned} \overline{u}_t + \overline{u\cdot\nabla u} - \nu\triangle\overline{u} + \nabla\overline{p} &= \overline{f} \text{ and} \\ \nabla\cdot\overline{u} &= 0. \end{aligned}$$

This is not closed since

$$\overline{u\cdot\nabla u}\,(=\overline{\nabla\cdot(u\,u)}\,) \neq \overline{u}\cdot\nabla\overline{u}\,(=\nabla\cdot(\overline{u}\,\overline{u})).$$

There are many closure models used in LES, see [S01], [J04], [BIL06] for a survey. We consider herein approximate deconvolution models. Approximate de-convolution models,

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studied herein, are used, with success, in many simulations of turbulent flows. They are among the most accurate of turbulence models and one of the few for which a mathematical confirmation of their effectiveness is possible. Briefly, an approximate deconvolution operator (constructed in section 2.1) denoted by D_N is an operator satisfying

$$\phi = D_N(\overline{\phi}) + O(\delta^{2N+2}) \quad \text{for smooth } \ \phi.$$

Since $D_N\overline{u}$ gives an approximation to u to accuracy $O(\delta^{2N+2})$ in the smooth flow regions it is justified to use it as a closure approximation which retains this formal order of accuracy. Doing so results in the family of Stoltz-Adams de-convolution model for the closure problem

$$\overline{uu} \simeq \overline{D_N\overline{u}D_N\overline{u}} + O(\delta^{2N+2}). \tag{1.3}$$

Using this closure approximation, the models introduced by Adams and Stolz are given by

$$w_t + \nabla \cdot (\overline{D_N w \ D_N w}) - \nu \triangle w + \nabla q + \chi\,(w - \overline{w}) = \overline{f}, \ \textit{and} \ \nabla \cdot w = 0. \tag{1.4}$$

1.1 Nomenclature

The nomenclature used is standard and defined where first used herein. We briefly give a summary of it next.

u, p : The true velocity and pressure, solutions of the Navier-Stokes equations (1.1).

w, q : The continuum velocity and pressure predicted by the LES model.

δ : The averaging radius of the filter used in the LES model.

$\widehat{\phi}$: The Fourier transform of the function ϕ for the Cauchy problem and the Fourier coefficient of ϕ for the periodic problem.

$\mathbf{k}, k$: The dual variable or wave number vector and wave number, respectively; $k = |\mathbf{k}| = (\mathbf{k}_1^2 + \mathbf{k}_2^2 + \mathbf{k}_3^2)^{\frac{1}{2}}$.

$||v||$: The L^2 norm of the indicated function.

$E(v)(t)$: The true, total kinetic energy of the indicated velocity field at time t:

$$E(v)(t) := \frac{1}{2L^3}||v(\cdot,t)||^2.$$

$E_{model}(v)(t)$: The kinetic energy of the LES model at time t, given by:

$$E_{model}(v)(t) := \frac{1}{2L^3}\{||v(\cdot,t)||_N^2 + \delta^2||\nabla v(\cdot,t)||_N^2\}.$$

$E(k)$: The distribution of the time averaged kinetic energy by wave number.

$E_{model}(k)$: The distribution by wave number of the LES time averaged model's kinetic energy.

$< \cdot >$: Time averaging.

$\varepsilon(v)(t)$: The (non-averaged) energy dissipation rate,

$$\varepsilon(v)(t) := \frac{\nu}{L^3}||\nabla v(\cdot,t)||^2.$$

ε : The time averaged energy dissipation rate of the true, Navier-Stokes velocity, $\varepsilon =< \varepsilon(v)(t) >$.

$\varepsilon_{model}(v)(t)$: The (non-averaged) LES model's energy dissipation rate, given by:

$$\varepsilon_{model}(v)(t) := \frac{\nu}{L^3}\{||\nabla v(\cdot,t)||_N^2 + \delta^2||\triangle v(\cdot,t)||_N^2\}.$$

ε_{model} : The time averaged energy dissipation rate of the LES model, $\varepsilon_{model} =< \varepsilon_{model}(v)(t) >$.

$P(v)(t)$: Power input

$$P(v)(t) := \frac{1}{L^3}(f(\cdot, t), v(\cdot, t)).$$

$P_{model}(v)(t)$: Power input of the LES model

$$P_{model}(v)(t) := \frac{1}{L^3}(f(\cdot, t), v(\cdot, t))_N.$$

α : The Kolmogorov constant, around 1.6.

Re : The Reynolds number.

ρ, μ, ν : Respectively, the fluids density, viscosity and kinematic viscosity.

U, L : The large scales characteristic velocity and length scale used to define the Reynolds number.

A : The differential operator that defines the differential filter, $Av := (-\delta^2\triangle + 1)v$.

D_N : The approximate deconvolution operator.

$\overline{v}$: Overbar denotes the average of the indicated function, $\overline{v} = A^{-1}v$.

η, η_{small} : The length scale of the smallest persistent eddies; the Kolmogorov micro-scale.

η_{model} : The model's micro-scale being the length scale of the model's smallest persistent eddies.

v_{small} : The velocity scale of the smallest persistent eddies in the model's solution.

$[\cdot]$: The units or dimensions.

Re_{small} : A Reynolds number based on the scales of the smallest persistent eddies.

2 A synopsis of K41 phenomenology

Turbulent flows consist of three dimensional eddies of various sizes. In 1941, I. Kolmogorov gave a remarkable, universal description of the eddies in turbulent flow by combining a judicious mix of physical insight, conjecture, mathematical analysis and dimensional analysis, e.g., Frisch [F95], Pope [P00]. In his description, the largest eddies are deterministic in nature. Those below a critical size are dominated by viscous forces, and die very quickly due to these forces. This critical length scale (the Kolmogorov micro-scale) is $\eta = O(Re^{-3/4})$[1] in $3d$. From this estimate, it follows that direct numerical simulation of a $3d$ flow thus requires $\Delta x = \Delta y = \Delta z = O(Re^{-3/4})$ giving $O(Re^{+9/4})$ mesh points in space per time step, and thus is often not computationally economical or even feasible. This estimate is based upon existence of an energy cascade in turbulent flow problems and Kolmogorov's above estimate of the micro-scale at the bottom of the energy cascade. Since this energy cascade theory is extended herein (and in other papers as well) beyond the Navier-Stokes equations, the answers to important questions about it must be reviewed.

Why do solutions of the Navier-Stokes equations (1.1) exhibit an energy cascade? And, should it be expected that solutions of (1.4) have their own energy cascade? The answer to the first question has been understood since the work of L. F. Richardson and I. Kolmogorov. We shall briefly review the answer (which is given also in Chapter 1 of most books on turbulence) because its answer also contains the answer to the second question (which we have developed in this report). The Navier-Stokes equations and their solutions have the following well-known features;

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[1] The length scale of the smallest persistent eddy is traditionally denoted by η rather than l .

- If $\nu = 0$ the total kinetic energy of the flow is exactly conserved[2]:

$$E(u)(t) = E(u)(0) + \int_0^t \int_\Omega f \cdot u d\mathbf{x} dt.$$

- The nonlinearity conserves energy globally (since $\int_\Omega u \cdot \nabla u \cdot u dx = 0$) but acts to transfer energy to smaller scales by breaking down eddies into smaller eddies (for example, if $u \simeq (U \sin(\frac{\pi x_1}{l}), 0, 0)^{tr}$ has wave length $2l$ and frequency $\frac{\pi}{l}$ then $u \cdot \nabla u \simeq \frac{U^2 \pi}{2l} (\sin(\frac{\pi x_1}{l/2}), 0, 0)^{tr}$ has shorter wave length l).

- If $\nu > 0$, them the viscous terms dissipate energy from the flow globally:

$$E(u)(t) + \int_0^t \varepsilon(u)(t')dt' = E(u)(0) + \int_0^t \int_\Omega f \cdot u d\mathbf{x} dt, \text{ where } \varepsilon(u)(t') \geq 0.$$

- For Re large the energy dissipation due to the viscous terms is negligible except on very small scales of motion. For example, if $u \simeq (U \sin(\frac{\pi x_1}{l}), 0, 0)^{tr}$ then

$$\begin{aligned} \text{viscous term on this scale} &= -\nu \triangle u \simeq \pi^2 \frac{\nu U}{l^2} (\sin(\frac{\pi x_1}{l}), 0, 0)^{tr}, \text{from which:} \\ \text{energy dissipation on this scale} &= \varepsilon(u) \simeq \frac{C}{L^3} \frac{\nu U^2}{l^2}. \end{aligned}$$

  Thus the nonlinear term dominates and the viscous term is negligible if

$$\frac{U^2}{l} >> \frac{\nu U}{l^2} \text{ , i.e., } \frac{lU}{\nu} >> 1.$$

- The forces driving the flow input energy persistently into the largest scales of motion.

The picture of the energy cascade that results from these effects is thus: *energy is input into the largest scales of the flow. There is an intermediate range in which nonlinearity drives this energy into smaller and smaller scales and conserves the global energy because dissipation is negligible. Eventually, at small enough scales dissipation is nonnegotiable and the energy in those smallest scales is driven to zero exponentially fast.* This is the physical reasoning behind Richardson's famous description:

> "Big whirls have little whirls
> That feed on their velocity,
> And little whirls have lesser whirls,
> And so on to viscosity."

Inspired by this description, in 1941 I. Kolmogorov gave a quantitative and universal characterization of the energy cascade (often called the K-41 theory). The most important components of the K-41 theory are the time (or ensemble) averaged energy dissipation rate, ε, and the distribution of the flows averaged kinetic energy across wave numbers, $E(k)$.

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[2] For the physical reasoning in this appendix and sections 4 it is perhaps appropriate to suppose that the energy equality holds and sidestep the deeper questions concerning weak vs. strong solutions and energy equality vs. energy inequality, e.g., [Ga00], [Gal94].

Given the velocity field of a particular flow, $u(\mathbf{x},t)$, the (time averaged) energy dissipation rate of that flow is defined to be

$$\varepsilon :=< \frac{1}{L^3}\int_\Omega \nu|\nabla u(\mathbf{x},t)|^2 d\mathbf{x} \ > . \tag{2.1}$$

Further, the K-41 theory states that at high enough Reynolds numbers there is a range of wave numbers

$$0 < k_{\min} := U\nu^{-1} \leq k \leq \varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}} =: k_{\max} < \infty, \tag{2.2}$$

known as the inertial range, beyond which the kinetic energy in u is negligible, and in this range

$$E(k) \doteq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{2.3}$$

where α is the universal Kolmogorov constant whose value is generally believed to be between 1.4 and 1.7 (for example, Wyngaard and Pao [WP72] found a value of $\alpha = 1.62$ in studies of atmospheric turbulence), k is the wave number and ε is the particular flow's energy dissipation rate. In this formula, the energy dissipation rate $< \varepsilon >$ is the only parameter which differs from one flow to another. Indeed, in Pope [P00], figure 6.14 page 235 in [P00], the power spectrums of 17 different turbulent flows taken from Saddoughi and Veeravalli [SV94] (which also contains the references to the particular experiments) are plotted on log-log plots. The slope of the linear region in this plot has the universal value of $-\frac{5}{3}$ for all 17 turbulent flows, exactly corresponding to the $k^{-\frac{5}{3}}$ law.

We review this argument of Kolmogorov, which is adapted in the next section. It begins with a physical conjecture that:

Conjecture 2.1. *The time averaged kinetic energy only depends on the time averaged energy dissipation rate ε and the wave number k .*

Beginning with this, postulate a simple power law dependency of the form

$$E(k) \simeq C\varepsilon^a k^b. \tag{2.4}$$

If this relation is to hold the units, denoted by $[\cdot]$ on the LHS must be the same as the units on the RHS, $[LHS] = [RHS]$. The three quantities in the above have the units

$$[k] = \frac{1}{length}, [\varepsilon] = \frac{length^2}{time^3}, [E(k)] = \frac{length^3}{time^2}.$$

Inserting these units into the above relation gives

$$\begin{aligned} \frac{length^3}{time^2} &= \frac{length^{2a}}{time^{3a}}\frac{1}{length^b} = length^{2a-b}time^{-3a}, \text{ giving} \\ 3a &= 2, 2a-b=3, \text{ or } a=\frac{2}{3}, b=-\frac{5}{3}. \end{aligned}$$

Thus, Kolmogorov's law follows

$$E(k) = \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \text{ over the inertial range } 0 < k \leq C(LRe^{-\frac{3}{4}})^{-1}.$$

The above estimate $\eta \sim LRe^{-\frac{3}{4}}$ for the Kolmogorov micro-scale is derived by similar physical reasoning. Let the reference large scale velocity and length (which are used in the definition of the Reynolds number) be denoted by U, L. At the scales of the smallest persistent eddies (the bottom of the inertial range) we shall denote the smallest scales of velocity and length by v_{small}, η. We form two Reynolds numbers:

$$Re = \frac{UL}{\nu}, Re_{small} = \frac{v_{small}\eta}{\nu}.$$

The global Reynolds number measures the relative size of viscosity on the large scales and when Re is large the effects of viscosity on the large scales are then negligible. The smallest scales Reynolds number similarly measures the relative size of viscosity on the smallest persistent scales. Since it is non-negligible we must have

$$Re_{small} \simeq 1, \text{ equivalently } \frac{v_{small}\eta}{\nu} \simeq 1.$$

Next comes an assumption of statistical equilibrium: Energy Input at large scales = Energy dissipation at smallest scales. The largest eddies have energy which scales like $O(U^2)$ and associated time scale $\tau = O(\frac{L}{U})$. The rate of energy transfer/energy input is thus $O(\frac{U^2}{\tau}) = O(\frac{U^3}{L})$[3]. The small scales energy dissipation from the viscous terms scales like

$$\varepsilon_{small} \simeq \nu |\nabla u_{small}|^2 \simeq \nu (\frac{v_{small}}{\eta})^2.$$

Thus we have the second ingredient:

$$\frac{U^3}{L} \simeq \nu (\frac{v_{small}}{\eta})^2.$$

Solving the first equation for v_{small} gives $v_{small} \simeq \frac{\nu}{\eta}$. Inserting this value for the small scales velocity into the second equation, solving for the length-scale η and rearranging the result in terms of the global Reynolds number gives the following estimate for η which determines the above estimate for the highest wave-number in the inertial range:

$$\eta = \eta_{Kolmogorov} \simeq Re^{-\frac{3}{4}} L.$$

This estimate for the size of the smallest persistent solution scales is the basis for the estimate of $O(Re^{\frac{9}{4}})$ mesh-points in space per time step for DNS of turbulent flows.

3 Summary of results

The questions considered are: Does the model itself have an energy cascade? If so, what are the specifics of the model's energy cascade? What insight does it give on how the model truncates the number of persistent scales in the model's solution? and What is the length-scale in the smallest persistent eddy in the model's solution?

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[3] It is known for many turbulent flows that, as predicted by K-41, ε scales like $\frac{U^3}{L}$. This estimate expresses statistical equilibrium in K-41 formalism, [F95], [Les97], [P00], [S84], [S98] and has been proven as an upper bound directly from the Navier Stokes equations without any assumptions of homogeneity or isotropy for turbulent flows in bounded domains driven by persistent shearing of a moving boundary, Constantin and Doering [CD92] , and Wang [W97], . The same estimate has been proven, Foias [F97], Doering and Foias [DF02], Childress, Kerswell and Gilbert [CKG01] (others have also contributed to this important theory as well), when the flow is driven by a persistent body force, the boundary conditions are periodic and the forcing acts on the largest modes/ largest scales.

By reviewing the reasoning of Richardson and Kolmogorov, we show that the model is expected to have an energy cascade. If we apply A_δ to the model (1.4) (with $\chi = 0$) it becomes:

$$\frac{\partial}{\partial t}\left[w - \delta^2 \triangle w\right] + D_N(w) \cdot \nabla D_N(w) - \nu\left[\triangle w - \delta^2 \triangle^2 w\right] + \nabla P = f\,, \quad \text{in}\Omega \times (0,T). \quad (3.1)$$

Since D_N is spectrally equivalent to the identity (uniformly in k, δ, nonuniformly in N) the nonlinear interaction $D_N(w) \cdot \nabla D_N(w)$ (like those in the NSE) will pump energy from large scales to small scales. The viscous terms in (3.1) will damp energy at the small scales (more strongly than in the NSE in fact). Lastly, when $\nu = 0$, $f \equiv 0$ the model's kinetic energy in (3.1) is exactly conserved

$$E_{model}(w)(t) = E_{model}(\overline{u}_0).$$

Thus, (3.1) satisfies all the requirements for the existence of a Richardson - like energy cascade for E_{model}.

By examining the details of the energy cascade of the model, we see a second mechanism for fast but not exponential truncation of the number of scales of the model's solution. Over the wave numbers corresponding to the resolved scales, $0 \leq k \leq \frac{1}{\delta}$, i.e. over length scales: $L \geq l \geq \delta$ we see that the model correctly predicts an energy spectrum of the form $\alpha_{\ model}\varepsilon_{model}^{2/3}k^{-5/3}$. Above the cutoff frequency and down to the model's micro-scale, the kinetic energy in the model's solution drops algebraically almost like k^{-4} according to $\alpha_{\ model}\varepsilon_{model}^{2/3}\delta^{-2}k^{-11/3}$. The model thus algebraically truncates the effective scales present. The derivation of these results involves the classical dimensional analysis arguments of Kolmogorov coupled with precise mathematical knowledge of the model's kinetic energy balance. Also the result is verified using the dynamical argument of Kraichnan, [K71].

The micro-scale of the model (the length-scale of the smallest persistent structure in the model's solution) is shown to be

$$\eta_{model} \simeq Re^{-\frac{3}{10}} L^{\frac{2}{5}} \delta^{\frac{3}{5}}.$$

This is typically smaller than the desired cutoff length-scale of $O(\delta)$ (see however section 3.2). Thus the behavior of the model in the intermediate range $\delta \geq l \geq \eta_{model}$ is critical. In fact it is easy to calculate that $\delta = \eta_{model} \ \Leftrightarrow \ \delta \simeq Re^{-3/4}$ and the flow is fully resolved.

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