System of First Order Differential Equations with Constant Coefficients

1. (Eigenproblem) Find an eigenpair for the matrix \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

2. (Fundamental Solution Matrix) The matrix \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) has an eigenpair \( \{i, \begin{bmatrix} 1 \\ i \end{bmatrix} \} \).

Find the fundamental solution matrix \( \Psi(t) \) to the differential system \( x' = Ax \) with the property that \( \Psi(0) \) is the identity matrix.

3. (Undetermined Coefficients) Find a particular solution to \( x' = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} x + e^{-t} \cos(2t) \begin{bmatrix} 6 \\ 2 \end{bmatrix} \).

4. (Variation of Parameters) Let \( A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \) and \( \Psi(t) = e^t \begin{bmatrix} 1 - t & t \\ -t & 1 + t \end{bmatrix} \). (a) Show that \( \Psi(t) \) is a fundamental solution matrix to \( \dot{y}' = Ay \). (b) Find the solution to the initial value problem \( x' = Ax + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

5. (Initial Value Problem) Solve \( x' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x \), \( x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

Second Order Linear Differential Equation with Constant Coefficients

1. (Homogeneous equations) Find the general solution to
   (a) \( u'' - 3u' + 2u = 0 \),  (b) \( u'' + u' + 2u = 0 \),  (c) \( u'' - 2u' + u = 0 \)

2. (Undetermined Coefficients) Find a particular solution to
   (a) \( u'' + 2u = 1 + 2t + t^2 \)  (b) \( u'' - 2u' + u = e^t \)  (c) \( u'' + u = 4\sin t \)

3. (Variation of Parameters) Find general solution to \( u'' + u = \sec t \)

4. (Initial Value Problem) Solve \( u'' + u = 2 \), \( u(0) = 0 \), \( u'(0) = 1 \).

Some Theoretical Proofs

1. Suppose that \( A \) is a real matrix and \( y(t) \) is a complex vector valued function solving \( y' = Ay \). Show that both the real and imaginary parts of \( y \) solve \( x' = Ax \).

2. Assume that \( A \) is a constant matrix and \( \Phi(t) \) is a solution matrix to \( x' = Ax \). Let \( \Psi(t) = \Phi'(t) \). Show that \( \Psi(t) \) is also a solution matrix to \( x' = Ax \).

3. Let \( L \) be a linear differential operator. Assume that \( L(u_1) = 0 \), \( L(u_2) = 0 \) and \( L(u_p) = f \). Show that for any constants \( c_1 \) and \( c_2 \), the function \( u = c_1 u_1 + c_2 u_2 + u_p \) solves \( L(u) = f \).

4. Let \( L(u) = u'' + pu' + qu \) where \( p \) and \( q \) are real functions. Assume that \( w = (t - it^2)e^{it} \) solves \( L(w) = te^{it} \). Find a particular solution to \( L(u) = t \cos t + 2t \sin t \).