MATH 0220 MIDTERM II REVIEW

1. **(Vector Algebra)** Let \( \mathbf{a} = (3, 4), \mathbf{b} = (-5, 12) \). Find \( |\mathbf{a}|, 2\mathbf{a} - 5\mathbf{b}, \mathbf{a} \cdot \mathbf{b} \) and the angle between \( \mathbf{a} \) and \( \mathbf{b} \). Also find the unit vector in the direction of \( \mathbf{a} \) and the projection of \( \mathbf{b} \) in the direction of \( \mathbf{a} \).

   **Solution.** \( |\mathbf{a}| = \sqrt{3^2 + 4^2} = 5 \), \( 2\mathbf{a} - 5\mathbf{b} = (2 \cdot 3 - 5 \cdot (-5), 2 \cdot 4 - 5 \cdot 12) = (31, -52) \), \( \mathbf{a} \cdot \mathbf{b} = 3 \cdot (-5) + 4 \cdot 12 = 33 \). The angle between \( \mathbf{a} \) and \( \mathbf{b} \) is \( \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \arccos \frac{33}{25} \). The unit vector in the direction of \( \mathbf{a} \) is \( \mathbf{e} = \frac{\mathbf{a}}{|\mathbf{a}|} = \left( \frac{3}{5}, \frac{4}{5} \right) \). The projection of \( \mathbf{a} \) in the direction of \( \mathbf{b} \) is \( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} = \frac{33}{25} \left( \frac{99}{25}, \frac{132}{25} \right) \).

2. **(Particle Motion)** A particle moving on a plane has position vector \( \mathbf{r}(t) = (1 + 2\cos t, 3\sin t) \) at time \( t \). (a) Find the average velocity during time interval \( [0, \pi] \). (b) Find the velocity vector, the speed, and the acceleration vector at time \( t = \pi/2 \). (c) Find a non-parametric equation describing the trajectory of the particle. (d) Find a parametric equation for the tangent line to the trajectory at \( t = \pi/2 \). (e) Find the slope of the tangent line in (d).

   **Solution.** (a) The average velocity is \( \frac{\text{displacement}}{\text{time used}} = \frac{\mathbf{r}(\pi) - \mathbf{r}(0)}{\pi - 0} = (-4/\pi, 0) \).

   (b) We compute \( \mathbf{r}'(t) = (-2\sin t, 3\cos t) \) and \( \mathbf{r}''(t) = (-2\cos t, -3\sin t) \). Hence, at \( t = \pi/2 \), the velocity vector is \( \mathbf{r}'(\pi/2) = (-2, 0) \), the speed is \( |\mathbf{r}'(\pi/2)| = \sqrt{(-2)^2 + 0^2} = 2 \), and the acceleration vector is \( \mathbf{r}''(\pi/2) = (0, -3) \).

   (c) In parametric form the trajectory can be written as \( x = 1 + 2\cos t \) and \( y = 3\sin t \), so that \( \cos t = (x - 1)/2 \) and \( \sin t = y/3 \). As \( \cos^2 t + \sin^2 t = 1 \), the non-parametric equation for the trajectory is \( (\frac{x-1}{2})^2 + (\frac{y}{3})^2 = 1 \), which is an ellipse.

   (d) At \( t = \pi/2 \), the particle is at \( \mathbf{r}(\pi/2) = (1, 3) \) and the tangent vector at the point is \( \mathbf{r}'(\pi/2) = (-2, 0) \). Hence, the vector equation of the tangent line is \( \mathbf{r} = \mathbf{r}(\pi/2) + \mathbf{r}'(\pi/2) s = (1, 3) + (-2, 0) s \). The parametric equation is \( x = 1 - 2s, y = 3 \) where \( -\infty < s < \infty \).

   (e) The slope is \( \frac{y'(t)}{x'(t)} \big|_{t=\pi/2} = \frac{3\cos t}{-2\sin t} \big|_{t=\pi/2} = 0 \).

3. **(Newton’s Law of Motion)** The engine of an aircraft of mass \( 3.6 \times 10^5 \) kg provides a \( 7.2 \times 10^5 \) N force. How long is a runway needed for the aircraft to reach its 100 m/s takeoff speed?

   **Solution.** From \( \mathbf{F} = \mathbf{ma} \) we have \( a = f/m = 2 \) (m/s²). Solving \( v' = a = 2 \) we obtain \( v = 2t + c \). Assuming \( v(0) = 0 \) we have \( c = 0 \) so \( v = 2t \). Solving \( s' = v = 2t \) we have \( s = t^2 + c \). Suppose \( s(0) = 0 \). We have \( s(t) = t^2 \) is the time \( T \) that the aircraft reaches its takeoff speed 100 m/s satisfies \( 2T = 100 \) or \( T = 50 \) (s). The distance travelled on the runway is \( s(T) = 50^2 = 2500 \) m. Thus, the runway needs at least 2500 m long.

4. **(Sketching Curves)** Consider the curve \( y = f(x) \) where \( f(x) = x^{-1}e^x, x \neq 0 \). (a) Find intervals where \( f \) is increasing or decreasing. (b) Find intervals where \( f \) is concave or convex. (c) Find any horizontal or vertical asymptotes. (d) Sketch the curve and mark all the points of local extremum and inflection.

   **Solution.** Computation gives \( f'(x) = x^{-1}e^x - x^{-2}e^x \) and \( f''(x) = x^{-1}e^x - x^{-2}e^x - x^{-2}e^x - 2x^{-3}e^x \).

   (a) Solving \( f'(x) = 0 \) gives \( x = 1 \). Thus (i) in \( (-\infty, 0) \) and in \( (1, \infty) \), \( f' > 0 \) and \( f \) decreases; (ii) in \( (0, 1) \), \( f' < 0 \) and \( f \) increases. (b) The equation \( f''(x) = 0 \) has no solution. Thus \( f'' > 0 \) and \( f \) is convex in \( (0, \infty) \); \( f'' < 0 \) and \( f \) is concave in \( (-\infty, 0) \). (c) \( \lim_{x \to 0^+} f(x) = \infty, \lim_{x \to 0^-} f(x) = -\infty \), \( \lim_{x \to \infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = 0 \). Thus, \( x = 0 \) is a vertical asymptote, and \( y = 0 \) is a horizontal asymptote (as \( x \to -\infty \)). (d) There is no inflection points, and there is only one local minimum, obtained at \( x = 1 \) with value \( f(1) = e \). The graph is omitted here.
5. (Linear Approximation and Newton’s Method) (a) By a tangent line approximation find $\sqrt{26}$.
(b) Via the Newton’s iteration find the positive root to $x^2 = 26$ (accurate upto $10^{-3}$).

Solution. (a) Set $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$. Since $\sqrt{26} = 5$ we take $a = 25$. The tangent line approximation is $f(x) \sim f(a) + f'(a)(x-a) = f(25) + f'(25)(x - 25) = 5 + \frac{1}{10}(x - 25)$. Thus $\sqrt{26} = f(26) \sim 5 + \frac{1}{10} = 5.1$

(b) We solve $f(x) = 0$ where $f(x) = x^2 - 26$. The Newton’s iteration is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 26}{2x_n}$. Set $x_0 = 5$. We have $x_1 = 5 - \frac{25 - 26}{10} = 5.1; x_2 = 5.1 - \frac{5.1^2 - 26}{10.2} = 5.1 - 0.0010$. Thus, the root $\approx 5.099$.
Remark: $5.099^2 > 26; \quad 5.098^2 < 26$. Thus $5.099 > \sqrt{26} > 5.098$.

6. (Optimization) (a) Find a point on the parabola $y = x^2 - 1/2$ that is closest to the point $(2, 0)$.

Solution. Suppose the point is $(x, y)$. Then $y = x^2 - 1/2$ and the distance from the point to $(2, 0)$ is $f(x) = \sqrt{(x - 2)^2 + (x^2 - 1/2 - 0)^2} = \sqrt{x^4 - 4x + 17/4}$. As $f'(x) = (4x^3 - 4)/(2\sqrt{x^4 - 4x + 17/4})$, solving $f'(x) = 0$ gives $x = 1$. Thus the point is $(1, 1/2)$ and the shortest distance from all points on the parabola to $(2, 0)$ is $\sqrt{5}/4$.

(b) A box with a square base, rectangular sides, and open top is to contain 6 ft^3 of space. If the cost of the material is $3/ft^2$ for its base and $2/ ft^2$ for its sides, determine its dimensions so that the cost of the material is a minimum.

Solution. Suppose the height is $h$ and the width is $x$. Then the volume of the box is $x^2h = 6$ so that $h = 6/x^2$. The total cost is $C(x) = 3[x^2] + 2*[4xh] = 3x^2 + 48/x$. As $C'(x) = 6x - 48/x^2$, solving $C'(x) = 0$ gives $x = 2$ (ft) and $h = 6/x^2 = 3/2$ (ft). Thus, the box has dimension $2 \times 2 \times 1.5$ measured in ft.

7. (Related Rates) (a) The volume of a spherical balloon is expanding at 36 m^3/h. How fast is the radius of the balloon increasing when the radius of the balloon is 3 m^3.

Solution. The volume $V$ and radius $R$ obey $V = \frac{4}{3}\pi R^3$, so $\frac{dV}{dt} = 4\pi R^2$ or $\frac{dR}{dt} = \frac{1}{4\pi R^2}$. The volume $V$ and time $t$ obey $\frac{dV}{dt} = 36$ (m^3/h). Hence, $\frac{dR}{dt} = \frac{dR}{dV} \frac{dV}{dt} = \frac{9}{\pi R^2}$. When $R = 3$, $\frac{dR}{dt} = 1/\pi$ (m/h).

(b) The width and length of a rectangular box increase at speeds 4 m/s and $-3$ m/s respectively. Find the rate of increasing of the area of the box when the width and length of the box are 3 (m) and 2m respectively.

Solution. The width $x$, the length $y$, the area $A$, and time $t$ obey $A = xy$, $\frac{dx}{dt} = 4$, and $\frac{dy}{dt} = -3$. Hence, $\frac{dA}{dt} = \frac{dx}{dt}y + xy\frac{dy}{dt} = 4y - 3x$. When $x = 3$ and $y = 2$, we have $\frac{dA}{dt} = 4*2 - 3*3 = -1$ m^2 /s; i.e., the area decreases at 1 m^2/s.

8. (L’Hôpital’s Rule) Find the limits: (a) $\lim_{x\to 0} \frac{e^x - 1}{x}$ (b) $\lim_{x\to 0} \frac{(1 + x)^{1/x}}{x}$ (c) $\lim_{x\to \infty} x \ln(1 + 1/x)$.

(a) The limit is of $\frac{0}{0}$ indeterminate form, so $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1$.

(b) Set $y = (1 + x)^{1/x}$. $\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1 + x)}{x}$ $= \lim_{x\to 0} \frac{1/(1 + x)}{x} = 1$. Thus $\lim_{x\to 0} (1 + x)^{1/x} = e$.

(c) Writing $x \ln(1 + 1/x) = \frac{\ln(1 + 1/x)}{1/x}$ the limit can be regarded as a $\frac{0}{0}$ indeterminate form. Hence $\lim_{x\to \infty} x \ln(1 + 1/x) = \lim_{x\to \infty} \frac{\ln(1 + 1/x)}{1/x} = \lim_{x\to \infty} (\ln(1 + 1/x))' = \lim_{x\to \infty} \frac{1}{1(x)'(1/x)' = 1}$. 

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