SOLUTIONS

Problem 1: (20 points) Solve the following initial value problem

\[ \frac{dy}{dx} = \frac{1 + 2x^2}{x} y, \quad y(1) = 1 \]

Solution
The equation is separable and can be written as

\[ \frac{dy}{y} = \frac{1 + 2x^2}{x} dx \]

which, after integrating both sides, becomes

\[ \int \frac{dy}{y} = \int \frac{1}{x} dx + \int 2x dx \]

\[ \ln|y| = \ln|x| + x^2 + C \]

\[ y = xe^{x^2+C} \]

Application of the initial condition \( y(1) = 1 \) gives \( C = -1 \) and hence the final answer is:

\[ y(x) = xe^{x^2-1} \]
Problem 2: (20 points) Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s², while air resistance provides deceleration \((0.001) \, v^2 \, \text{ft/s}^2\) when the car’s velocity is \(v\) feet per second.

(a) Find the car’s maximum possible (limiting) velocity

(b) Find how long it takes the car to attain 90% of its limiting velocity.

(Note that \(\tanh^{-1}(0.9) \approx 1.47\).)

Solution

According to the statement of the problem the velocity \(v = v(t)\) of the car obeys the following differential equation:

\[
\frac{dv}{dt} = 10 - (0.001)v^2, \quad v(0) = 0
\]

(a) The limiting velocity of the car is the stable fixed point of this differential equation. The fixed point is given by the solution of the equation \(10 - (0.001)v^2 = 0\) and is \(v = v_\tau = 100 \, \text{ft/s}\).

(b) The differential equation is separable and can be written as

\[
\frac{dv}{10 - (0.001)v^2} = dt
\]

which, after integrating both sides, becomes

\[
0.1 \int \frac{dv}{1 - (0.0001)v^2} = t + C
\]

\[
10 \tanh^{-1}(0.01v) = t + C
\]

\[
v = 100 \tanh\left(\frac{t}{10} + C_1\right)
\]

Application of the initial condition \(v(0) = 0\) gives \(C_1 = 0\), and hence

\[
v(t) = 100 \tanh\left(\frac{t}{10}\right)
\]

The car will attain 90% of its limiting velocity at the time \(t\) that obeys the equation \(v(t) = 0.9v_\tau = 90 \, \text{ft/s}\). Thus

\[
t = 10 \tanh^{-1}(0.9) = 14.7 \, \text{s}
\]
**Problem 3:** (20 points) Find a solution of the initial value problem

\[ 2x^2y - x^3y' = y^3 \quad y(1) = 1/2 \]

**Solution**

After rewriting the equation so that \( y' \) appears as an explicit term we obtain

\[ y' = 2 \frac{y}{x} - \frac{y^3}{x^2} \]

We recognize that the equation is homogeneous. We use a substitution \( v = y/x \), which implies

\[ y = vx, \quad \frac{dy}{dx} = x \frac{dv}{dx} + v \]

and rewrite the differential equation in the form

\[ x \frac{dv}{dx} + v = 2v - v^3 \]

\[ \frac{dv}{dx} = \frac{v-v^3}{x} \]

This differential equation is separable and can be solved as follows

\[ \int \frac{dv}{v} + \int \frac{vdv}{v-1} = \int \frac{dx}{x} \]

\[ \ln|v| - \frac{1}{2} \ln|v-1| = \ln|x| + C \]

\[ \frac{v}{\sqrt{1-v^2}} = Kx \]

\[ v^2 = K^2x^2(1-v^2) \]

By backsubstituting \( v = y/x \) we obtain

\[ y^2 = K^2x^2(x^2 - y^2) \]

\[ y^2 = \frac{K^2x^4}{1+K^2x^2} \]

Finally, the application of the initial condition \( y(1) = 1/2 \) implies \( K^2 = 1/3 \) and the solution is

\[ y^2 = \frac{x^4}{3+x^2} \]
Alternative solution

After rewriting the equation so that \( y' \) appears as an explicit term we obtain

\[
y' - 2 \frac{y}{x} = -\frac{y^3}{x^3}
\]

We recognize a Bernoulli equation with \( P(x) = -(2/x) \), \( Q(x) = -x^{-3} \), and \( n = 3 \). Hence we substitute \( v = y^{1-n} = y^{-2} \) which implies

\[
y = v^{-1/2}, \quad \frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}
\]

Substitution from the differential equation for \( dy/dx \) yields

\[
\frac{dv}{dx} = -2y^{-3} \left( 2 \frac{y}{x} - \frac{y^3}{x^3} \right)
\]

\[
\frac{dv}{dx} + 4 \frac{v}{x} = \frac{2}{x^3}
\]

We recognize a linear differential equation for \( v = v(x) \) with \( P(x) = 4/x \) and \( Q(x) = 2x^{-3} \). The integrating factor is

\[
\rho = \exp \left( 4 \int \frac{1}{x} \, dx \right) = \exp(4 \ln|x|) = x^4
\]

Multiplication of the differential equation for \( v = v(x) \) by \( \rho \) yields

\[
\frac{d}{dx} (x^4 v) = 2x
\]

\[
x^4 v = x^2 + C
\]

\[
v = \frac{x^2 + C}{x^4}
\]

Backsubstitution \( v = y^{-2} \) implies that the general solution is

\[
y^{-2} = \frac{x^4}{C + x^2}
\]

Finally, the application of the initial condition \( y(1) = 1/2 \) implies \( C = 3 \) and the solution is

\[
y^2 = \frac{x^4}{3 + x^2}
\]
Problem 4: (20 points) Find the general solution of the differential equation

\[ xy' - 12x^4y^{2/3} = 6y \]

Solution

After rewriting the equation so that \( y' \) appears as an explicit term we obtain

\[ y' - \frac{6}{x}y = 12x^3y^{2/3} \]

We recognize a Bernoulli equation with \( P(x) = -(6/x) \), \( Q(x) = 12x^3 \), and \( n = 2/3 \). Hence we substitute \( v = y^{1-n} = y^{1/3} \) which implies

\[ y = v^3, \quad \frac{dy}{dx} = \frac{1}{3}y^{-2/3} \frac{dv}{dx} \]

Substituting from the differential equation for \( dy/dx \) gives

\[ \frac{dv}{dx} = \frac{1}{3}y^{-2/3} \left( \frac{6}{x}y + 12x^3y^{2/3} \right) = \frac{2}{x}y^{1/3} + 4x^3 \]

\[ \frac{dv}{dx} - \frac{2}{x}v = 4x^3 \]

We recognize a linear differential equation for \( v = v(x) \) with \( P(x) = -(2/x) \) and \( Q(x) = 4x^3 \). The integrating factor is

\[ \rho = \exp \left( -\int \frac{2}{x} \, dx \right) = \exp \left( -2 \ln |x| \right) = x^{-2} \]

Multiplication of the differential equation for \( v = v(x) \) by \( \rho \) yields

\[ \frac{d}{dx} \left( x^{-2}v \right) = 4x \]

\[ x^{-2}v = 2x^2 + C \]

\[ v = 2x^4 + Cx^2 \]

Backsubstitution \( y = v^3 \) implies that the general solution is

\[ y = (2x^4 + Cx^2)^3 \]
Problem 5: (20 points) Consider a breed of rabbits whose birth and death rates, $\beta$ and $\delta$, are each proportional to the rabbit population $P = P(t)$ with $\beta > \delta$.

(a) Show that

$$P(t) = \frac{P_0}{1 - kP_0 t}$$

where $k$ is a constant and $P_0 = P(0)$. Note that $P(t) \to \infty$ as $t \to 1/(kP_0)$. This is the doomsday.

(b) If $P_0 = 6$ and there are 9 rabbits after 10 months, when does doomsday occur?

Solution

(a) The population of rabbits obeys the differential equation $dP/dt = (\beta - \delta)P$. According to the statement of the problem $\beta = \beta_0 P$, $\delta = \delta_0 P$ which implies that

$$\frac{dP}{dt} = kP^2$$

where $k = \beta_0 - \delta_0$ is a constant. This separable equation can be solved as follows

$$\frac{dP}{P^2} = kdt$$

$$-\frac{1}{P} = kt + C$$

$$P = \frac{-1}{C + kt}$$

The assumption $P(0) = P_0$ implies that $C = -(1/P_0)$ and hence

$$P = \frac{P_0}{1 - kP_0 t}$$

(b) The assumptions $P_0 = 6$ and $P(10) = 9$ imply

$$9 = \frac{6}{1 - 60k}$$

$$k = 1/180$$

Hence the doomsday occurs when $t = 1/(kP_0) = 180/6 = 30$, i.e., after 30 months.