Problem 1:
Let $P$ be a projector. Show that the complementary projector $I - P$ projects onto $\text{null}(P)$.

SOLUTION:
The vector $v$ is in $\text{null}(P)$ if and only if $Pv = 0$, in which case $(I - P)v = v - Pv = v$, which is equivalent to saying that $v$ is in $\text{range}(I - P)$.

Problem 2:
Let $B$ be a $3 \times 3$ matrix. Each of the following operations on $B$,
1. doubling of column 1
2. halving row 3
3. adding row 3 to row 1
4. interchanging columns 2 and 1
can be expressed as a multiplication of $B$ by a matrix $E_i$. Find such matrices $E_i$. Write the result of application of all four operations on $B$ as a product of three matrices $ABC$.

SOLUTION:
\[
E_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
ABC = E_3E_2BE_1E_4 = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Problem 3:
Show that if $Q$ is orthogonal and upper triangular then it is diagonal, with each element on the diagonal being either 1 or $-1$.

SOLUTION:
The matrix $Q$ has the following form:
The condition of orthogonality, $Q^TQ = I$, implies:

$$q_i^T q_1 = (q_{11})^2 = 1 \text{ and hence } q_{11} = \pm 1$$

$$q_i^T q_k = q_{11}q_{1k} = 0 \text{ and hence } q_{1k} = \pm 1 \text{ for } k = 2, 3, \ldots, n$$

Thus, the first row of $Q$ is equal to $[1 0 0 \ldots 0]$.

By repeating the argument for rows 2, 3, ..., $n$ we obtain the result.

**Problem 4:**

Consider the matrix $A = I + uv^T$. Show that if $A$ is invertible, its inverse is given by $A^{-1} = I + \alpha uv^T$. Find the value of $\alpha$. What condition do the vectors $u$ and $v$ satisfy when $A$ is singular?

**SOLUTION:**

If $A$ is invertible the product of $I + \alpha uv^T$ with $A$ becomes

$$(I + \alpha uv^T)A = (I + \alpha uv^T)(I + uv^T) = I + (1 + \alpha)uv^T + \alpha uv^T uv^T = I + (1 + \alpha + \alpha \beta)uv^T$$

where $\beta = v^T u$. Hence, if

$$\alpha = \frac{-1}{1+\beta} = \frac{-1}{1 + v^T u}$$

then $I + \alpha uv^T$ is the inverse of $A$. Such a value of $\alpha$ exists only if $v^T u \neq -1$. In the opposite case, i.e., when $v^T u = -1$, $A$ is singular.

**Problem 5:**

Let $P$ be the orthogonal projector onto the span of $p$ and $q$, where $p$ and $q$ are orthonormal vectors. Show that $I - P$ is the product of two projectors, $I - pp^T$ and $I - qq^T$.

**SOLUTION:**

The orthonormality of $p$ and $q$ implies $p^T p = q^T q = 1$ and $p^T q = 0$. The orthogonal projector onto the span of $p$ and $q$ is the projector $P = UU^T$ where $U$ is a matrix with columns $p$ and $q$, i.e., $U = \begin{bmatrix} p & q \end{bmatrix}$. We will show that the result of the operation of the projector $I - P$ on an arbitrary vector $a$ is the same as a result of the application of the projectors $I - pp^T$ and $I - qq^T$.
\[(I - P)a = (I - UU^T)a = a - U \begin{bmatrix} p^T a \\ q^T a \end{bmatrix} = a - pp^T a - qq^T a\]
\[(I - pp^T)(I - qq^T)a = (I - pp^T - qq^T + pp^T qq^T)a = a - pp^T a - qq^T a\]

The two results are the same.

**Problem 6:**
Find an orthogonal projector \(P\) that projects onto \(\text{null}(A)\) where
\[
A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix}
\]

**SOLUTION:**
To determine \(\text{null}(A)\) we must find all solutions \(x\) of the system \(Ax = 0\) (by any method, e.g. Gauss elimination). The solutions form a 1D subspace spanned by the unit vector
\[
v = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \]
Thus,
\[
P = vv^T = \frac{1}{6} \begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}.
\]

**Problem 7:**
Calculate the Householder reflector that takes the vector \(\begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}^T\) into \(\begin{bmatrix} \sqrt{6} & 0 & 0 & 0 \end{bmatrix}^T\).

**SOLUTION:**
The Householder reflector is defined as \(H = I - 2\frac{vv^T}{v^Tv}\), where \(v\) is the vector connecting the original vector \(x\) and its reflection \(\|x\|e_1\). Here \(x = \begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}^T\),
\[\|x\|e_1 = \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 \end{bmatrix}^T, \quad v = \|x\|e_1 - x = \begin{bmatrix} \sqrt{6} - 1 & 0 & -2 & 1 \end{bmatrix}^T,\]
and hence
\[
H = I - \frac{1}{6 - \sqrt{6}} \begin{bmatrix} 7 - 2\sqrt{6} & 0 & 2(1 - \sqrt{6}) & \sqrt{6} - 1 \\ 0 & 0 & 0 & 0 \\ 2(1 - \sqrt{6}) & 0 & 4 & -2 \\ \sqrt{6} - 1 & 0 & -2 & 1 \end{bmatrix} = \frac{1}{6 - \sqrt{6}} \begin{bmatrix} \sqrt{6} - 1 & 0 & 2(\sqrt{6} - 1) & 1 - \sqrt{6} \\ 0 & 6 - \sqrt{6} & 0 & 0 \\ 2(\sqrt{6} - 1) & 0 & 2 - \sqrt{6} & 2 \\ 1 - \sqrt{6} & 0 & 2 & 5 - \sqrt{6} \end{bmatrix}
\]
**Problem 8:**

Find the distance of the point \( b = [1 \ 1 \ 0 \ 2]^T \) from the hyperplane defined as the span of \([1 \ 0 \ 2 \ 4]^T\) and \([0 \ -2 \ 1 \ 1]^T\), using least squares minimization.

**SOLUTION:**

Put \( a_1 = [1 \ 0 \ 2 \ 4]^T \) and \( a_2 = [0 \ -2 \ 1 \ 1]^T \). A point \( x \) in the span of \( a_1 \) and \( a_2 \) can be written as \( x = c_1 a_1 + c_2 a_2 = A c \) where \( A \) is a 4x2 matrix with columns \( a_1 \) and \( a_2 \):

\[
A = \begin{bmatrix}
1 & 0 \\
0 & -2 \\
2 & 1 \\
4 & 1 \\
\end{bmatrix}
\]

and \( c \) is a column vector with components \( c_1 \) and \( c_2 \). The distance of \( b \) from such a point \( x \) is equal to \( \|x - b\| \). Our goal is to find the point \( x \) in the span of \( a_1 \) and \( a_2 \) that is closest to \( b \), i.e., to find \( c \) that minimizes \( \|A c - b\| \).

The method of solving this least squares minimization problem calls for finding the reduced QR factorization of \( A \), i.e., \( A = Q \hat{R} \), calculating the orthogonal projection \( y \) of \( b \) onto \( \text{range}(A) \) as \( y = Q^T b \) and then solving the system of equations \( \hat{R} c = y \) :

**QR factorization:**

\[
\begin{align*}
r_{11} &= \|a_1\| = \sqrt{21}, & q_1 &= \frac{a_1}{r_{11}} = \frac{1}{\sqrt{21}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, & r_{12} &= q_1^T a_2 = \frac{6}{\sqrt{21}}, \\
v_2 &= a_2 - r_{12} q_1 = \frac{1}{7} \begin{bmatrix} -2 \\ -14 \\ 3 \\ -1 \end{bmatrix}, & r_{22} &= \|v_2\| = \frac{\sqrt{210}}{7}, & q_2 &= \frac{a_2}{r_{22}} = \frac{1}{\sqrt{210}} \begin{bmatrix} -2 \\ -14 \\ 3 \\ -1 \end{bmatrix} \\
y &= \hat{Q}^T b = \begin{bmatrix} 9 \\ \frac{\sqrt{21}}{\sqrt{210}} \\ -18 \\ \sqrt{210} \end{bmatrix},
\end{align*}
\]
The solution of $\hat{R}c = \mathbf{y}$, i.e.,

$$
\begin{bmatrix}
\sqrt{21} & 6 \\
\sqrt{21} & \sqrt{21} \\
0 & 7
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 
\begin{bmatrix}
9 \\
-18 \\
\sqrt{21}
\end{bmatrix}
$$

is $c = [3/5 \quad -3/5]$. The distance of $\mathbf{b}$ from the hyperplane is

$$
\|A\mathbf{c} - \mathbf{b}\| = \left\|egin{array}{c}
-2 \\
1 \\
3 \\
-1
\end{array}\right\| = \frac{3}{\sqrt{5}}.
$$

**Problem 9:**
Compute the QR factorization of the following matrix

$$
A = 
\begin{bmatrix}
-1 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -1
\end{bmatrix}
$$

**SOLUTION:**

\begin{align*}
& r_{11} = \|a_1\| = \sqrt{2}, \quad q_1 = \frac{a_1}{r_{11}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad r_{12} = q_1^T a_2 = -\frac{3}{\sqrt{2}}, \quad v_2 = a_2 - r_{12} q_1 = \frac{1}{2} \begin{bmatrix} -1 \\ \frac{1}{2} \\ 2 \end{bmatrix}, \\
& r_{22} = \|v_2\| = \frac{\sqrt{6}}{2}, \quad q_2 = \frac{v_2}{r_{22}} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad r_{13} = q_1^T a_3 = \frac{1}{\sqrt{2}}, \quad r_{23} = q_2^T a_3 = -\frac{3}{\sqrt{6}}, \\
& v_3 = a_3 - r_{13} q_1 - r_{23} q_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad r_{33} = \|v_3\| = 0.
\end{align*}

The vector $q_3$ cannot be determined from $v_3$. To complete the QR factorization we must determine $q_3$ as a unit vector orthogonal to both $q_1$ and $q_2$. Either by inspection or by solution of the following system of equations: $q_1^T q_3 = 0$, $q_2^T q_3 = 0$, $q_3^T q_3 = 1$, we
find that $q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The solution is $Q = \begin{bmatrix} -1 & -1 & 1 \\ \sqrt{2} & \sqrt{6} & \sqrt{3} \\ 1 & -1 & 1 \\ \sqrt{2} & \sqrt{6} & \sqrt{3} \\ 0 & 2 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} \sqrt{2} & -3 & 1 \\ \sqrt{2} & \sqrt{6} & 0 \\ \sqrt{6} & \sqrt{3} & -3 \end{bmatrix}$.

**Problem 10:**
Using QR factorization find the solution of the system of equations $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

**SOLUTION:**
The first step of the solution procedure is to find the QR factorization of $A$. The second step is to compute the vector $y$ as $y = Q^T b$. The third step is to solve the system of equations $R \hat{x} = y$ by back substitution.

QR factorization:

$r_1 = \|a_1\| = \sqrt{14}$, $q_1 = \frac{a_1}{r_1} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $r_{12} = q_1^T a_2 = \frac{20}{\sqrt{14}}$, $v_2 = a_2 - r_{12} q_1 = \frac{1}{7} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

$r_{22} = \|v_2\| = \sqrt{\frac{21}{7}}$, $q_2 = \frac{v_2}{r_{22}} = \frac{1}{\sqrt{\frac{21}{7}}} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$, $r_{13} = q_1^T a_3 = \frac{29}{\sqrt{14}}$, $r_{23} = q_2^T a_3 = \frac{4}{\sqrt{21}}$

$v_3 = a_3 - r_{13} q_1 - r_{23} q_2 = \frac{1}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $r_{33} = \|v_3\| = \frac{1}{\sqrt{6}}$, $q_3 = \frac{v_3}{r_{33}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$y = \hat{Q}^T b = \begin{bmatrix} 19 \\ \frac{1}{\sqrt{14}} \\ -1 \\ \frac{1}{\sqrt{21}} \\ -1 \\ \frac{1}{\sqrt{6}} \end{bmatrix}$
The solution of $\dot{\mathbf{R}} \mathbf{e} = \mathbf{y}$, i.e.,

$$
\begin{bmatrix}
\sqrt{14} & 20 & 29 \\
\sqrt{14} & \sqrt{14} & \sqrt{14} \\
0 & \sqrt{21} & 4 \\
0 & 7 & \sqrt{21} \\
0 & 0 & 1 \\
\end{bmatrix}
\mathbf{x} =
\begin{bmatrix}
19 \\
\sqrt{14} \\
-1 \\
\sqrt{21} \\
-1 \\
\sqrt{6} \\
\end{bmatrix}
$$

is $\mathbf{x} = [2 \ 1 \ -1]^T$. 