Problem 1:
Determine the relative condition number for the following mathematical problems:

a) \( f(x, y) = \frac{x^T y}{\|x\| \|y\|} \)

b) \( f(x) = 1 + \tan(x) \)

Problem 2:
Determine whether the following algorithms are backward stable, stable, or unstable. For backward stable algorithms determine their accuracy.

a) Computation of \( f(x) = (1 + x)^3 \) as \( \tilde{f}(x) = (1 \oplus \text{fl}(x)) \odot (1 \oplus \text{fl}(x)) \odot (1 \oplus \text{fl}(x)) \)

b) Computation of \( \sqrt{x} \) starting with an initial guess \( y \) as one step of Newton’s iteration: \( \tilde{f}(x, y) = \left( [\text{fl}(x) \odot \text{fl}(y)] \oplus \text{fl}(y) \right) \odot 2 \)

a) Computation of one root of the polynomial \( x^2 - 2x + c \) in terms of \( c \) for \( c \approx 1 \):
\( f(c) = 1 + \sqrt{1 - c} \) as \( \tilde{f}(c) = 1 \oplus \sqrt{1 \oplus \text{fl}(c)} \).
(Hint: Assume \( c = 1 + d \epsilon_{\text{machine}} \). Neglect terms of order \( O(\epsilon_{\text{machine}}^2) \).) Use \( \sqrt{1 + \epsilon_1} = 1 + \epsilon_2 / 2 \) where \( |\epsilon_1|, |\epsilon_2| \leq \epsilon_{\text{machine}} \)

Problem 3:
Determine the accuracy of the following algorithm for calculation of \( f(x, y) = \frac{x - y}{x + y} \):
\( \tilde{f}(x, y) = (\text{fl}(x) \odot \text{fl}(y)) \odot (\text{fl}(x) \oplus \text{fl}(y)) \)

Problem 4:
Solve the following system of equations by LU factorization (with pivoting if necessary)

\[
egin{align*}
-4x_1 - 4x_2 - 3x_3 - x_4 &= 1 \\
-7x_1 - 7x_2 + 7x_3 + 6x_4 &= -6 \\
-x_1 + x_3 + x_4 &= -2 \\
3x_1 + 3x_2 + x_3 + 2x_4 &= -2
\end{align*}
\]
Problem 5:
Let $A$ and $B$ be a nonsingular lower triangular $m \times m$ matrix. Show that the diagonal elements of $A$ are all nonzero. Show that $A^T$ is lower triangular.

Problem 6:
Compute the LU factorization with partial pivoting, (i.e., find $P$, $L$, $U$ such that $PA = LU$) for the following matrix

$$A = \begin{bmatrix} -2 & 6 & 6 & 6 \\ 8 & 0 & -10 & 0 \\ -1 & -6 & 3 & 4 \\ -2 & 3 & -3 & 2 \end{bmatrix}$$

Problem 7:
Solve the following systems of equations by Cholesky factorization (if the coefficient matrix is positive definite) or by Gaussian elimination (otherwise)

a)

$$\begin{align*}
1x_1 - 2x_2 + 3x_3 - x_4 &= 10 \\
-2x_1 + 8x_2 - 6x_3 + 4x_4 &= -30 \\
3x_1 - 6x_2 + 10x_3 - 4x_4 &= 30 \\
x_1 + 4x_2 - 4x_3 + 7x_4 &= 11
\end{align*}$$

b)

$$\begin{align*}
3x_1 + 3x_2 - 5x_4 &= 2 \\
3x_1 + 2x_2 + 2x_3 + 3x_4 &= 2 \\
2x_2 + 4x_3 + 3x_4 &= 5 \\
-5x_1 + 3x_2 + 3x_3 + 2x_4 &= -4
\end{align*}$$

Problem 8:
Show that if $A$ has a full rank then $A^T A$ is positive definite.
Problem 9:
Calculate LU factorization without pivoting for the following matrices.

a) \[
\begin{bmatrix}
4 & 2 & 1 \\
8 & 4 & 1 \\
-1 & -1 & 1
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
3 & 2 & 3 & 4 \\
5 & 2 & 3 & 1 \\
1 & 1 & 3 & -2 \\
0 & 3 & 1 & 1
\end{bmatrix}
\]