Problem 1:
Each of the following statements is either true or false. Give a proof for those that are true and give a counterexample for those that are false:
a) If $\lambda$ is an eigenvalue of $A$ then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.
b) If $A$ is real and $\lambda$ is an eigenvalue of $A$ then $-\lambda$ is also an eigenvalue of $A$.
c) If all the eigenvalues of $A$ are zero, then $A = 0$.

SOLUTION:
a) The statement is true. $\lambda$ is an eigenvalue of $A$ if and only if there is $x \neq 0$ such that $Ax = \lambda x$. It follows that 
$$(A - \mu I)x = Ax - \mu x = \lambda x - \mu x = (\lambda - \mu)x$$
and hence $\lambda - \mu$ is an eigenvalue of $A - \mu I$.
b) The statement is false. Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This matrix is real and it has eigenvalues $\lambda_1 = \lambda_2 = 1$ but no eigenvalue $-\lambda_1 = -1$.
c) The statement is false. Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This matrix has eigenvalues $\lambda_1 = \lambda_2 = 0$ but $A$ is not equal to zero matrix.

Problem 2:
Find the eigenvalue decomposition $A = XAX^{-1}$ of the following matrix:

$$A = \begin{bmatrix} -1 & 0 & -4 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 3 & -1 \end{bmatrix}$$

SOLUTION:
In order to calculate the decomposition we must first determine the eigenvalues and eigenvectors of $A$. 
The characteristic polynomial is
\[
\det(A - \lambda I) = \begin{vmatrix} -1 - \lambda & 0 & -4 & 0 \\ 0 & 2 - \lambda & -1 & 0 \\ 0 & 0 & 3 - \lambda & 0 \\ 0 & 3 & 3 & -1 - \lambda \end{vmatrix} = (-1 - \lambda)(2 - \lambda)(3 - \lambda)(3 - \lambda)
\]
which has the roots 3, 2, and -1. Thus the eigenvalues are \( \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = -1 \).

For \( \lambda_1 = 3 \) the eigenvector is obtained by solving
\[
(A - \lambda_1 I)x_1 = \begin{bmatrix} -4 & 0 & -4 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & -4 \end{bmatrix} x_1 = 0 \text{ which yields, for example, } x_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}
\]

For \( \lambda_2 = 2 \) the eigenvector is obtained by solving
\[
(A - \lambda_2 I)x_2 = \begin{bmatrix} -3 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 3 & -3 \end{bmatrix} x_2 = 0 \text{ which yields, for example, } x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

The eigenvalue \( \lambda_3 = -1 \) is a double eigenvalue and we expect two linearly independent eigenvector obtained by solving
\[
(A - \lambda_3 I)x_3 = \begin{bmatrix} 0 & 0 & -4 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} x_3 = 0 \text{ which yields } x_3 = \begin{bmatrix} s \\ 0 \\ 0 \\ t \end{bmatrix}
\]

with \( s \) and \( t \) free parameters. We can pick, for example, \( x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \) and \( x_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

The columns of the matrix \( \mathbf{X} \) are the eigenvectors of \( \mathbf{A} \) and the diagonal elements of \( \mathbf{\Lambda} \) are the corresponding eigenvalues. Thus
\[
\mathbf{X} = \begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

**Problem 3:**
Find the orthogonal diagonalization \( \mathbf{A} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^T \) for the following symmetric matrix \( \mathbf{A} \):
\[
\mathbf{A} = \begin{bmatrix}
4 & -2 & 1 \\
-2 & 4 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

**SOLUTION:**
In order to calculate the decomposition we must first determine the eigenvalues and eigenvectors of \( \mathbf{A} \).

The characteristic polynomial is
\[
\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix}
4 - \lambda & -2 & 1 \\
-2 & 4 - \lambda & 1 \\
1 & 1 & 1 - \lambda
\end{vmatrix} = (4 - \lambda) \begin{vmatrix}
4 - \lambda & 1 \\
1 & 1 - \lambda
\end{vmatrix} - (-2) \begin{vmatrix}
-2 & 1 \\
1 & 1 - \lambda
\end{vmatrix} + (-2) \begin{vmatrix}
4 - \lambda & 1 \\
1 & 1
\end{vmatrix}
\]
\[
= (4 - \lambda)((4 - \lambda)(1 - \lambda) - 1) + 2(-2(1 - \lambda) - 1) + (-2 - (4 - \lambda))
\]
\[
= -\lambda (\lambda^2 - 9\lambda + 18)
\]
\[
= -\lambda (\lambda - 3)(\lambda - 6)
\]
which has the roots 6, 3, and 0. Thus the eigenvalues are \( \lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 0 \).

For \( \lambda_1 = 6 \) the eigenvector is obtained by solving
\[
(\mathbf{A} - \lambda_1 \mathbf{I}) \mathbf{x}_1 = \begin{bmatrix}
-2 & -2 & 1 \\
-2 & -2 & 1 \\
1 & 1 & -5
\end{bmatrix} \mathbf{x}_1 = \mathbf{0}
\]
which yields, for example, \( \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \)

For \( \lambda_2 = 3 \) the eigenvector is obtained by solving
\[
(\mathbf{A} - \lambda_2 \mathbf{I}) \mathbf{x}_2 = \begin{bmatrix}
1 & -2 & 1 \\
-2 & 1 & 1 \\
1 & 1 & -2
\end{bmatrix} \mathbf{x}_2 = \mathbf{0}
\]
which yields, for example, \( \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)
For $\lambda_3 = 0$ the eigenvector is obtained by solving

$$(A - \lambda_3 I)x_3 = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

which yields, for example, $x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

The columns $q_1, q_2, q_3$ of the matrix $Q$ are the normalized eigenvectors, i.e.,

$$q_i = \frac{x_i}{\|x_i\|}$$

The result is

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$