Problem 1:
Consider the matrix
\[
Q = \begin{bmatrix}
1/2 & 0 & \sqrt{3}/2 \\
0 & 1 & 0 \\
-\sqrt{3}/2 & 0 & 1/2
\end{bmatrix}
\]
Show that \(Q\) is orthogonal. What transformation of \(\mathbb{R}^3\) does it correspond to?
(Hint: Find the vector \(a\) that is invariant under \(Q\). Pick a vector \(b\) orthogonal to \(a\). Find the angle \(\alpha\) between \(b\) and \(Qb\). If this angle is independent of the choice of \(b\), then \(Q\) corresponds to a rotation about \(a\) by the angle \(\alpha\). Think about other possibilities.)

SOLUTION:
\[
Q^TQ = \begin{bmatrix}
1/2 & 0 & -\sqrt{3}/2 \\
0 & 1 & 0 \\
\sqrt{3}/2 & 0 & 1/2
\end{bmatrix}
\begin{bmatrix}
1/2 & 0 & \sqrt{3}/2 \\
0 & 1 & 0 \\
-\sqrt{3}/2 & 0 & 1/2
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix} = I
\]
Hence \(Q\) is orthogonal.
Since \(Q\) is orthogonal, the mapping \(x \rightarrow Qx\) corresponds to either rotation or reflection of \(\mathbb{R}^3\).
Let us find the vectors \(a\) that are invariant under \(Q\):
\[
Qa = a
\]
\[
(Q - I)a = 0
\]
\[
\begin{bmatrix}
-1/2 & 0 & \sqrt{3}/2 \\
0 & 0 & 0 \\
-\sqrt{3}/2 & 0 & -1/2
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}
\]
The solution is \(a = [0 \ 1 \ 0]^T\) or its scalar multiples. Therefore, \(Q\) corresponds to either a reflection through the \(y\)-axis or a rotation about that axis.
To confirm or rule out the rotation we pick a vector \(b\) orthogonal to \(a\) and find the angle \(\alpha\) between \(b\) and \(Qb\). A vector \(b\) is orthogonal to \(a\) if and only if
\[ \mathbf{b}^T \mathbf{a} = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 1 & 0 \end{bmatrix} = b_2 = 0. \]

Thus, any vector \( \mathbf{b} = \begin{bmatrix} b_1 & 0 & b_3 \end{bmatrix} \) is orthogonal to \( \mathbf{a} \). The angle \( \alpha \) between \( \mathbf{b} \) and \( \mathbf{Qb} \) is given by the formula

\[
\cos \alpha = \frac{\mathbf{b}^T (\mathbf{Qb})}{\|\mathbf{b}\|\|\mathbf{Qb}\|}.
\]

Because \( \|\mathbf{b}\| = \sqrt{b_1^2 + b_3^2} \) and, by orthogonality of \( \mathbf{Q} \), \( \|\mathbf{Qb}\| = \|\mathbf{b}\| \), we have

\[
\cos \alpha = \frac{1}{b_1^2 + b_3^2} \begin{bmatrix} b_1 & 0 & b_3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix} = \frac{1}{2}.
\]

Thus, the angle between \( \mathbf{b} \) and \( \mathbf{Qb} \) is 60º regardless of the choice of \( \mathbf{b} \) (provided \( \mathbf{b} \) is orthogonal to \( \mathbf{a} \)) and \( \mathbf{Q} \) corresponds to the rotation about the y-axis by 60º.

**Problem 2:**

Find the 3×3 orthogonal matrix that corresponds to the reflection of \( \mathbb{R}^3 \) through the plane \( x + y = 0 \).

**SOLUTION:**

A reflection of \( \mathbb{R}^3 \) through the plane \( x + y = 0 \) takes the vector \( \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T \) into the vector \( \begin{bmatrix} -v_2 & -v_1 & v_3 \end{bmatrix}^T \) (see the Figure). Thus the problem calls for finding the orthogonal matrix \( \mathbf{Q} \) such that

\[
\mathbf{Q} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_2 \\ -v_1 \\ v_3 \end{bmatrix}.
\]

Clearly, the only orthogonal matrix \( \mathbf{Q} \) obeying this equation for all \( v_i \) is

\[
\mathbf{Q} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
Problem 3:
Find the orthogonal projector \( P \) onto range(A) where
\[
A = \begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 
\end{bmatrix}
\]
What is null(P)? What is the image under P of the vector \([1 \ 2 \ 4]^T\)?

SOLUTION:
A formula given in the lecture tells us that if \( U \) is a matrix with orthonormal columns then the
matrix \( P = UU^T \) is an orthogonal projector which range is equal to range(U). The columns of
the matrix \( A \) are orthogonal but not orthonormal and hence we cannot simply put \( P = AA^T \). Instead
we take the matrix \( U \) that contains the normalized columns of \( A \):
\[
U = \left[ \frac{a_1}{\|a_1\|} \mid \frac{a_2}{\|a_2\|} \right] = \begin{bmatrix}
1/\sqrt{2} & 0 \\
1/\sqrt{2} & 0 \\
0 & 1 
\end{bmatrix}
\]
It follows that range(U) = range(A) and therefore \( P \) is given by
\[
P = UU^T = \begin{bmatrix}
1/\sqrt{2} & 0 \\
1/\sqrt{2} & 0 \\
0 & 1 
\end{bmatrix}\begin{bmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & 0 & 1 
\end{bmatrix} = \begin{bmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1 
\end{bmatrix}.
\]
Since \( P \) is a projector,
\[
\text{null}(P) = \text{range}(I - P) = \text{span} \left[ \begin{bmatrix}
1/2 \\
-1/2 \\
0 
\end{bmatrix}, \begin{bmatrix}
-1/2 \\
1/2 \\
0 
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
0 
\end{bmatrix} \right] = \text{span} \left[ \begin{bmatrix}
1 \\
-1 \\
0 
\end{bmatrix} \right].
\]
The image under \( P \) of the vector \([1 \ 2 \ 4]^T\) is
\[
P = \begin{bmatrix}
1 \\
2 \\
4 
\end{bmatrix} = \begin{bmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1 
\end{bmatrix}\begin{bmatrix}
1 \\
1 \\
4 
\end{bmatrix} = \begin{bmatrix}
3/2 \\
3/2 \\
4 
\end{bmatrix}.