Problem 1:
Let $Q$ and $R$ be the QR factors of a symmetric tridiagonal matrix $H$. Show that the product $K = RQ$ is again a symmetric tridiagonal matrix.

(Hint: Prove the symmetry of $K$. Show that $Q$ has Hessenberg form and that the product of an upper triangular matrix and a Hessenberg matrix is again a Hessenberg matrix. Then use the symmetry of $K$.)

**SOLUTION:**

Symmetry of $K$:

From the symmetry of $H$ it follows that $QR = H = HT = R^TQ^T$. Using this relation we find that

$$K = RQ = Q^TRQ = Q^TR^TQ = Q^TR^T = K^T$$

and hence $K$ is symmetric.

Hessenberg form of $Q$:

The $m \times m$ matrix $H$ is tridiagonal and hence it has a Hessenberg form which implies that $h_{ij} = 0$ whenever $i > j + 1$. The matrix $R$ of the QR factorization of $H$ is upper triangular and hence $r_{ij} = 0$ whenever $i > j$. Suppose $i > j + 1$. We have

$$h_{ij} = \sum_{k=1}^{m} q_{ik} r_{kj} = q_{i1} r_{1j} + q_{i2} r_{2j} + ... + q_{ij} r_{jj}$$

When $j = 1$ the relation above has only one term on the right hand side, i.e., $h_{i1} = q_{i1} r_{11}$. Because $r_{11} \neq 0$ we can conclude that $q_{i1} = 0$ whenever $h_{i1} = 0$, i.e., whenever $i > 2$.

When $j = 2$ we have $h_{i2} = q_{i1} r_{12} + q_{i2} r_{22}$ which, for $i > 2$, reduces to $h_{i2} = q_{i2} r_{22}$. Thus, again, $q_{i2} = 0$ whenever $h_{i2} = 0$, i.e., whenever $i > 3$. By continuing this argument we see that the matrix $Q$ has zeros below the first subdiagonal just as $H$ does and hence $Q$ has Hessenberg form.

Hessenberg form of $K$:

Similarly as above, the matrix $R$ is upper triangular with $r_{ij} = 0$ whenever $i > j$ and $Q$ is Hessenberg with $q_{ij} = 0$ whenever $i > j + 1$ and hence

$$h_{ij} = \sum_{k=1}^{m} r_{ik} q_{kj} = \sum_{k=i}^{j+1} r_{ik} q_{kj}$$

It follows that when $i > j + 1$ the sum in the relation above has no terms and hence $k_{ij} = 0$. Thus, $K$ is a Hessenberg matrix.

Because $K$ is both Hessenberg and symmetric, it follows that $K$ is tridiagonal.
Problem 2:

Determine one eigenvalue of the following matrix using Rayleigh Quotient iteration, starting with initial guess \( \mathbf{v}^{(0)} = [0\ 1]^T \). Terminate iteration after 3 steps, i.e., after you obtain \( \lambda^{(3)} \). What is the approximate eigenvector \( \mathbf{v}^{(3)} \)? What is the error of \( \lambda^{(3)} \)?

\[
\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}
\]

**SOLUTION:**

\( \mathbf{v}^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( \lambda^{(0)} = (\mathbf{v}^{(0)})^T \mathbf{A} \mathbf{v}^{(0)} = 6 \)

First iteration:

The solution of \( (\mathbf{A} - \lambda^{(0)} \mathbf{I}) \mathbf{w} = \begin{bmatrix} -3 & -2 \\ -2 & 0 \end{bmatrix} \mathbf{w} = \mathbf{v}^{(0)} \) is \( \mathbf{w} = \begin{bmatrix} -1/2 \\ 3/4 \end{bmatrix} \)

\( \mathbf{v}^{(1)} = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\sqrt{13}} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -0.5547 \\ 0.8321 \end{bmatrix} \)

\( \lambda^{(1)} = (\mathbf{v}^{(1)})^T \mathbf{A} \mathbf{v}^{(1)} = \frac{90}{13} = 6.9231 \)

Second iteration:

The solution of \( (\mathbf{A} - \lambda^{(1)} \mathbf{I}) \mathbf{w} = \begin{bmatrix} -51/13 & -2 \\ -2 & -12/13 \end{bmatrix} \mathbf{w} = \mathbf{v}^{(1)} \) is \( \mathbf{w} = \frac{1}{64 \sqrt{13}} \begin{bmatrix} -1326 \\ 2665 \end{bmatrix} \)

\( \mathbf{v}^{(1)} = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{13 \sqrt{52429}} \begin{bmatrix} -1326 \\ 2665 \end{bmatrix} = \begin{bmatrix} -0.4455 \\ 0.8953 \end{bmatrix} \)

\( \lambda^{(2)} = (\mathbf{v}^{(2)})^T \mathbf{A} \mathbf{v}^{(2)} = \frac{367002}{52429} = 6.9999809 \)

Third iteration:

Solution of \( (\mathbf{A} - \lambda^{(2)} \mathbf{I}) \mathbf{w} = \begin{bmatrix} -3.9999809 & -2 \\ -2 & -0.9999809 \end{bmatrix} \mathbf{w} = \mathbf{v}^{(2)} \) is \( \mathbf{w} = \begin{bmatrix} -23446.917 \\ 46893.834 \end{bmatrix} \)

\( \mathbf{v}^{(3)} = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \begin{bmatrix} -0.44721360 \\ 0.89442719 \end{bmatrix} \)

\( \lambda^{(3)} = (\mathbf{v}^{(3)})^T \mathbf{A} \mathbf{v}^{(3)} = 7 \)

The eigenvalues of \( \mathbf{A} \) are solutions of

\[
\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3 - \lambda & -2 \\ -2 & 6 - \lambda \end{vmatrix} = \lambda^2 - 9 \lambda + 14 = (\lambda - 7)(\lambda - 2) = 0
\]

i.e., \( \lambda_1 = 7, \ \lambda_2 = 2 \). The error of \( \lambda^{(3)} \) is smaller than the machine accuracy \( \varepsilon_{\text{machine}} = 10^{-16} \).
Problem 3:
Perform the first two steps of the QR algorithm (i.e., compute $A^{(2)}$ and $\tilde{Q}^{(2)}$) for the following matrix. How close are the diagonal elements of $A^{(2)}$ to the eigenvalues of $A$?

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

SOLUTION:
Put $A^{(0)} = A$.

Compute QR factorization of $A^{(0)}$: $Q^{(1)} R^{(1)} = A^{(0)}$,

$$Q^{(1)} = \begin{bmatrix} -0.8944 & -0.3586 & 0.2673 \\ 0.4472 & -0.7171 & 0.5345 \\ 0 & 0.5976 & 0.8018 \end{bmatrix}, \quad R^{(1)} = \begin{bmatrix} -2.2361 & 1.7889 & -0.4472 \\ 0 & -1.6733 & 1.9124 \\ 0 & 0 & 1.0690 \end{bmatrix}$$

$$A^{(1)} = R^{(1)} Q^{(1)} = \begin{bmatrix} 2.8 & -0.7843 & 0 \\ -0.7843 & 2.3429 & 0.6389 \\ 0 & 0.6389 & 0.8571 \end{bmatrix}$$

Compute QR factorization of $A^{(1)}$: $Q^{(2)} R^{(2)} = A^{(1)}$,

$$Q^{(2)} = \begin{bmatrix} -0.9661 & -0.2467 & -0.0761 \\ 0.2582 & -0.9231 & -0.2849 \\ 0 & -0.2949 & 0.9555 \end{bmatrix}, \quad R^{(2)} = \begin{bmatrix} -2.8983 & 1.3279 & 0.1650 \\ 0 & -2.1665 & -0.8425 \\ 0 & 0 & 0.6370 \end{bmatrix}$$

$$A^{(2)} = R^{(2)} Q^{(2)} = \begin{bmatrix} 3.1429 & -0.5594 & 0 \\ -0.5594 & 2.2484 & -0.1878 \\ 0 & -0.1878 & 0.6087 \end{bmatrix}$$

The matrix $\tilde{Q}^{(2)}$ is given by

$$\tilde{Q}^{(2)} = Q^{(1)} Q^{(2)} = \begin{bmatrix} 0.7715 & 0.4729 & 0.4256 \\ -0.6172 & 0.3941 & 0.6810 \\ 0.1543 & 0.7881 & 0.5959 \end{bmatrix}$$

The eigenvalues of $A$ are solutions of

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 2 - \lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 10\lambda + 4 = -(\lambda - 2)(\lambda^2 - 4\lambda + 2) = 0$$

i.e., $\lambda_1 = 2 + \sqrt{2}$, $\lambda_2 = 2$, $\lambda_3 = 2 - \sqrt{2}$. The difference between the diagonal elements of $A^{(2)}$ and the eigenvalues of $A$ is $\text{diag}(A^{(2)}) - [\lambda_1 \lambda_2 \lambda_3] = [-0.2714 \ 0.2484 \ 0.0229]$. 
Computer Assignment 6:

a) Write a MATLAB function \([Q,L]=qralg(A)\) that computes the eigenvalues and eigenvectors of a square, symmetric \(m \times m\) matrix \(A\) using QR algorithm. The output variables are the \(m \times m\) orthogonal matrix \(Q\) which columns are the eigenvectors of \(A\) and \(m \times m\) matrix \(L\) that has the corresponding eigenvalues of \(A\) on the main diagonal. The program should terminate iteration when the norm of the offdiagonal elements of \(A^{(k)}\), i.e., \(\text{norm}(A-\text{diag(diag}(A)))\), is smaller than \(10^{-6}\).

b) Use the function \(qralg\) to calculate the eigenvalues and eigenvectors of

\[
A = \begin{bmatrix}
2 & 6 & 4 & -4 & -5 & -10 \\
6 & 12 & -2 & 9 & 5 & 9 \\
4 & -2 & 0 & -1 & -3 & 14 \\
-4 & 9 & -1 & 14 & -6 & 8 \\
-5 & 5 & -3 & -6 & 2 & 8 \\
-10 & 9 & 14 & 8 & 8 & 8 \\
\end{bmatrix}
\]

Record the number of iterations needed to achieve the desired accuracy.

PROGRAM:

```matlab
function [Q,L]=qralg(A)
m = length(A);
QQ = eye(m);
k = 0;
D = 1;
while D > 1e-6
    [Q,R] = qr(A);
    A = R*Q;
    QQ = QQ*Q;
    D = norm(A - diag(diag(A)));
    k = k+1;
end
Q = QQ;
L = A;
disp(k)
```
OUTPUT:

```matlab
>> A = [2 6 4 -4 -5 -10; 6 12 -2 9 5 9; 4 -2 0 -1 -3 14; -4 9 -1 14 -6 8; -5 5 -3 -6 2 8; -10 9 14 8 8 8]
A =
2 6 4 -4 -5 -10
6 12 -2 9 5 9
4 -2 0 -1 -3 14
-4 9 -1 14 -6 8
-5 5 -3 -6 2 8
-10 9 14 8 8 8

>> [Q, A] = qralg(A)

Q =
-0.1821 -0.4101 0.3863 0.3839 -0.4722 -0.5282
0.4982 0.2399 0.3805 0.6289 0.3914 0.0275
0.1903 0.5935 -0.2860 -0.1038 -0.1111 -0.7118
0.5226 0.0858 0.5175 -0.5036 -0.4217 0.1426
0.1590 0.1875 -0.4073 0.4363 -0.6505 0.4003
0.6197 -0.6161 -0.4364 -0.0481 0.1031 -0.1817

A =
31.2723 0.0000 -0.0000 -0.0000 -0.0000 -0.0000
0.0000 -19.1968 0.0000 -0.0000 0.0000 0.0000
-0.0000 0.0000 16.1596 0.0000 -0.0000 -0.0000
0.0000 -0.0000 -0.0000 11.5658 0.0000 0.0000
0.0000 -0.0000 -0.0000 0.0000 -10.3079 -0.0000
-0.0000 0.0000 -0.0000 0.0000 -0.0000 8.5070

There was 145 iterations required to reach the accuracy 10^{-6}.
```